Density of integral points on varieties:
Mordell orbifold conjecture and special varieties

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ABSTRACT

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Marta Benozzo

The objective of this thesis is to understand F.Campana's conjectures about density of integral points on varieties defined over a number field. In the case of curves this is solved by Siegel and Faltings' theorems: the degree of the canonical divisor of a curve determines whether it has finitely or infinitely many rational points (possibly after a finite extension of the base field). In higher dimension, varieties can be neither potentially dense (the set of rational points becomes dense after at most a finite extension of the base field), nor mordellic (rational points are always finitely many in an open dense subset). The conjectures studied in the thesis address the problems of characterizing potential density and mordellicity of a variety and of constructing a (unique) fibration which splits each variety in its mordellic part, the base of the fibration, and potentially dense part, the fibers. To deal with this problem, F.Campana introduces the notion of orbifold pair: to each variety, one can attach an orbifold divisor, which allows keeping track of multiple fibers of fibrations. Using this tool, he was able to construct a fibration, the core map, which has (orbifold) special general fiber, conjectured to be the potentially dense varieties, and base of (orbifold) general type, conjectured to be the mordellic ones.

Contents

I wandered lonely as a cloud

I wandered lonely as a cloud
That floats on high o'er vales and hills,
When all at once I saw a crowd,
A host, of golden daffodils;
Beside the lake, beneath the trees,
Fluttering and dancing in the breeze.

Continuous as the stars that shine
And twinkle on the milky way,
They stretched in never-ending line
Along the margin of a bay:
Ten thousand saw I at a glance,
Tossing their heads in sprightly dance.

The waves beside them danced; but they
Out-did the sparkling waves in glee:
A poet could not but be gay,
In such a jocund company:
I gazed—and gazed—but little thought
What wealth the show to me had brought:

For oft, when on my couch I lie
In vacant or in pensive mood,
They flash upon that inward eye
Which is the bliss of solitude;
And then my heart with pleasure fills,
And dances with the daffodils.

∼ *William Wordsworth*

Introduction

A problem that arises in Diophantine geometry is whether rational and integral points on a variety defined over a number field are dense or not. In particular, the goal would be to find geometric conditions that can characterize potential density (i.e. the closed rational points become Zariski dense after at most a finite extension of the base field) and mordellicity (i.e. there exists an open subset of the variety that contains only finitely many rational points, even after finite extensions of the base field).
Integral points can be interpreted as points that do not become points at infinity of the variety reduced modulo every prime of the ring of integers. This concept can be generalized introducing the notion of an orbifold pair, a variety together with a simple normal crossing divisor in the form $\Delta = \sum_{j=1}^{s}(1 - 1/m_j)D_j$, where D_j are prime (Weil) divisors. This is not completely new, it generalizes log-pairs that arise naturally in arithmetic geometry and play an important role in the minimal model program and in the study of moduli spaces of higher dimensional varieties.
The integral points of the orbifold pair are then the ones that, modulo all but finitely many primes, if they belong to any of the divisors D_j's, their intersection number with them is "higher than m_j", in a sense that is made precise in the first chapter. In particular, if all multiplicities are infinite, these are the integral points on the quasi-projective variety complement of the support of the divisor Δ; on the other hand, if Δ is the zero divisor, we are considering exactly the rational points. According to Campana, the problem of potential density of rational and integral points cannot be solved without taking into account also the "intermediate" cases.

The density problem was first formulated by Mordell in the case of projective curves (1922) and solved by Siegel (1929) in the affine case. Later, Faltings proved the projective case (1983). They understood that the geometrical invariant that characterizes density is the genus of a curve, or, equivalently, the degree of its canonical divisor: curves are potentially dense if and only if the degree of the canonical divisor is non-positive and they are mordellic elsewhere (see theorems 1.2.1 and 1.2.2).
In the orbifold case, the problem is still open, but there are many results towards the proof. Also in this case, it is conjectured that positivity of the degree of the canonical divisor of the orbifold pair (defined as the canonical divisor of the curve plus the divisor Δ) characterizes potential density as above. In particular, this is proven in the "classical version" 1.4 by reducing to Faltings theorem by means of suitable ramified covers. To finish the proof of the "general version" it is possible, using the previous results, to reduce

the situation to the study of just a few orbifold pairs. In this cases it is even possible to show that *abc* conjecture implies the result [4, section 3.6].

Thanks to the birational classification of surfaces made by the Italian school in the 19^{th} century, also in the case of surfaces a lot of progress has been made. The situation about rational points is almost understood and, as for integral and orbifold integral points, there are many results and examples that agree with the conjectures made for higher dimensional varieties. Anyway, the work is not completed yet, in many cases it is still open how to solve the conjectures.

In higher dimension the problem is pretty much open. The guiding ideas to approach it come from the fact that we expect that density properties are preserved by birational morphisms, Chevalley–Weil theorem 1.3.2 that says that they are preserved also by étale covers and Bombieri, Lang and Vojta's conjectures that say that varieties of general type are mordellic.
Also in the higher dimensional case, the canonical divisor plays a central role. Indeed, it is conjectured that the Kodaira dimension of a variety can characterize density properties. Thus, tools from birational geometry come into play.
Harris and Tschinkel recently introduced the notion of weakly special varieties, varieties X such that, if $X' \to X$ is an étale cover, then X' does not admit any dominant map towards a variety of general type. They conjecture that these are exactly the potentially dense varieties. Campana was able to construct a map that divides the weakly special part of a variety and the part of general type: the weak core map. It is a fibration from the variety X to a variety of general type, the weak core, with the property that the fibers are weakly special.
The main flaw of this approach is that the weak core is not preserved under étale covers. However, this problem can be solved by taking into account possible multiple fibers of fibrations and this is possible by considering orbifold pairs and "orbifold morphisms". This is also the main reason why we need to introduce orbifolds in the discussion.
The subsequent modification of the notion of weakly special varieties is the notion of special varieties (and special orbifold pairs), conjectured by Campana to be the potentially dense ones. The weak core map is modified into the core map, a fibration with special fibers and base of (orbifold) general type. It turns out, according to Campana's point of view, that there are three building blocks that can be distinguished using the invariant given by the Kodaira dimension:

- varieties (and orbifold pairs) X of (orbifold) general type, i.e. whose Kodaira dimension $\kappa(X) = \dim X$;
- varieties (and orbifold pairs) with 0 Kodaira dimension;
- rationally connected varieties (and orbifold pairs) characterized by the fact that they do not admit any fibration towards a variety with positive Kodaira dimension.

Special varieties are conjectured to be towers of fibrations over a point with general fibers of one of the last two types.

Now, we come to the structure of the thesis.

In the first chapter, we present the definitions of orbifold pairs and orbifold integral points.
Then, we discuss the known results for curves and surfaces. In particular, we study a proof of Siegel's theorem and of the classical version of Mordell orbifold conjecture.
This latter can be reduced to the study of some particular cases and the main idea to deal with them is to construct suitable orbifold étale covers from curves for which we already know density properties using Faltings' theorem. With a modification of Chevalley–Weil theorem for the orbifold case, we are then able to transfer the density result to our initial orbifold pair. Various techniques are used to construct these maps, including Riemann existence theorem and origami covers.
Siegel's theorem is studied using the approach of Corvaja and Zannier (2002) that reduced it to an application of subspace theorem 1.3.1.
The advantage of this approach is that it can be generalized to surfaces, but introducing some more hypotheses involving numerical conditions. This is discussed in the last section of this chapter 1.5, after recalling some general numerical properties. The last section contains also the birational classification of surfaces with the known results on rational points for each class.

To be able to study the higher dimensional case, we need some tools that are developed in the second chapter. In particular, in the first section we discuss some classical results about invertible sheaves, linear systems, how they can define maps to a projective space and when these maps are embeddings (ampleness properties). Besides, we see some useful formulas involving the canonical sheaf.
This first section allows us to have the right background to understand the definition of the Iitaka and the Kodaira dimensions, notions discussed in the second section. If L is a line bundle on a variety X, the Iitaka dimension $\kappa(X, L)$ is an invariant that gives the dimension of the image of our variety X under the rational morphism defined by mL for big m, it gives the asymptotic behavior of these maps. The Kodaira dimension is the same quantity, with $L = \omega_X$, the canonical sheaf.
An important property of these dimensions is the easy additivity property, which states that, if $p : X \to Z$ is a fibration and L is an invertible sheaf on X, then $\kappa(X, L) \leq \kappa(X_z, L_{|X_z}) + \dim Z$, where X_z is the general fiber of p. This theorem is studied in the third section.
The last two sections of the chapter are devoted to the construction of two important fibrations: the Iitaka–Moishezon fibration and the maximal rationally connected (MRC) quotient. The first one, given a line bundle L on a variety X, is birationally equivalent to all maps defined by mL for m big enough and sufficiently divisible. Furthermore, its general fibers have 0 Iitaka dimension.
On the other hand, the latter one is a fibration such that the very general fibers correspond to rationally connected components and the base is not uniruled. These maps are particularly important in our discussion because the weak core map (and the core map with suitable orbifold modifications) can be obtained by applying them alternately a finite number of times.

In the last chapter we see the constructions of the weak core and the core maps and their relation to conjectures about density properties.
The first section contains some technical results of Chow space theory and the construction of the $\mathcal{C}$-quotient, where $\mathcal{C}$ is a class of varieties with a given property. This map is a fibration with base belonging to the class and general fiber that does not admit any fibration towards a variety in the class $\mathcal{C}$.
This construction is then applied to varieties of general type to construct the weak core map, in the next section. Here, we present the notion of weakly special variety and an example that shows the necessity of taking into account multiple fibers and, therefore, pass to an orbifold point of view.
The third section develops the technical tools needed to deal with orbifolds. In particular, it is introduced the notion of orbifold base of a fibration, with which we keep track of multiple fibers, and we restate some definitions and results in the orbifold case.
At last, we are able to construct the core map, prove that the core is preserved under étale covers and state the main conjecture presented in the thesis, formulated by Campana in [4], which says that the core map splits every variety in its mordellic (the base of orbifold general type) and potentially dense part (the special fibers).

Notations

k number field
$\mathcal{O}_k$ ring of integers of k
$\mathcal{O}_{k,\mathfrak{p}}$ local ring at $\mathfrak{p}$, prime of $\mathcal{O}_k$
S finite set of places of k
$\mathcal{O}_{k,S}$ localization of $\mathcal{O}_k$ at S
$\mathbb{F}_\mathfrak{p}$ residue field at a prime $\mathfrak{p}$
$\mathcal{O}_X$ structure sheaf of a scheme X
$\mathcal{O}_{X,P}$ stalk of $\mathcal{O}_X$ at the point P
$X(k) = \{\mathrm{Spec}(k) \to X\}$ set of closed k-points of a scheme X
$\bar{K}$ algebraic closure of a field K
$\equiv_n$ congruence modulo n
$\mathrm{disc}(k/\mathbb{Q})$ discriminant of the number field k
$k_\mathfrak{p}$ completion of k at a prime $\mathfrak{p}$
R_I localization of a ring at the ideal I
$\pi_1(X)$ fundamental group of X
$H^0(X, L)$ global sections of the sheaf L on X
$h^0(X, L)$ dimension of the space of global sections of an $\mathcal{O}_X$-module L as a vector space over the field of definition of X
$H^i(X, L)$ i^{th} cohomology group associated with the sheaf L on X
$h^i(X, L)$ dimension of the space $H^i(X, L)$ of an $\mathcal{O}_X$-module L as a vector space over the field of definition of X
$V(I)$ zero locus of the ideal I
K_X canonical divisor of X
ω_X canonical bundle on X
$\mathfrak{m}_P$ maximal ideal of the local ring $\mathscr{F}_P$ for a sheaf $\mathscr{F}$ on a scheme X and a point $P \in X$
$\overline{X}$ (Zariski) closure of X

A **variety** in the whole thesis will refer to a separated irreducible and reduced algebraic noetherian scheme of finite type over a field k.

A property true for $m \gg 0$ means that there exists $m_0 > 0$ such that the property holds for all $m \geq m_0$

Given an irreducible variety X, we say that a property holds at a **general** point if it holds for all points in the complement of a Zariski closed subset (proper subset). A property holds at a **very general** point if it is satisfied off the union of countably many closed subvarieties (proper subvarieties).

Chapter 1

Orbifold integral points: curves and surfaces

In this chapter we discuss the notion of orbifold integral points over a number field, a generalization of integral and rational points. We, then, take a look at the situation on curves and surfaces. A classical result by Faltings relates the density of these points to the genus of the curve, in the general case the degree of the canonical divisor plus the orbifold divisor is conjectured to determine their behaviour. This result is known in the quasi-projective case thanks to Siegel's theorem and is completely proven in the "classical" setting. As for surfaces, thanks to their classification, much is known in the projective case. For the quasi-projective case there is a generalization of Siegel's theorem, which, however, does not characterize completely potential density and mordellicity as for curves. For the general orbifold case there are conjectures that agree with those that will be stated later for higher dimensional varieties, proven in many examples.
In the first section we discuss the definition of an orbifold pair, which is a variety together with a particular $\mathbb{Q}$-divisor, and the notion of orbifold integral points. To define these points we impose a condition on their intersection with the orbifold divisor modulo primes. We allow only "big enough" intersections.
The second section is devoted to the study of density of integral points on curves, here we present the known results and the orbifold Mordell conjecture.
Then, we present a proof of Siegel's theorem following the approach of Corvaja and Zannier, using subspace theorem 1.3.1.
The fourth section deals with the proof of the "classical" form of Mordell orbifold conjecture. The idea is to construct covers ramified in the support of the orbifold divisor and to study the relation between the rational points of the domain (for which we can use Faltings' theorem) and the orbifold integral points of the base of these covers.
The proof we saw for Siegel's theorem can be applied also to the case of surfaces, but adding some more numerical conditions. In the fifth section, after presenting the known results on rational and integral points on surfaces, we study some numerical properties and the adapted proof in this situation.

1.1 Orbifold integral points

This first sections is devoted to the definition of orbifold integral points. First, we need to be able to talk about "reduction modulo primes" of varieties and points over a number field, so we introduce the notion of models. Then, we discuss the notions of orbifold pair and orbifold integral points. We conclude this section with some easy examples of orbifold integral points of the projective line.

We start with a couple of examples to motivate the definitions.

Example 1.1.1.

- Let $k \subseteq \mathbb{C}$ be a number field, and $\mathcal{O}_k$ its ring of integers. Consider the projective line $\mathbb{P}^1(\mathbb{C}) = \{[x_0 : x_1] \,|x_0, x_1 \in \mathbb{C}\}$, then the rational points over k are $\mathbb{P}^1(k) = \{[x_0 : x_1] \,|x_0, x_1 \in k\}$, while the integral points are $\mathbb{P}^1(\mathcal{O}_k) = \{[x_0 : x_1] \,|x_0, x_1 \in \mathcal{O}_k\}$. In this case the integral and the rational points actually coincide as for every $x \in k$ we can find $n \in \mathbb{Z}$ such that $nx \in \mathcal{O}_k$, so if $[x_0 : x_1] \in \mathbb{P}^1(k)$, then there exists $n \in \mathbb{Z}$ for which $[x_0 : x_1] = [nx_0 : nx_1] \in \mathbb{P}^1(\mathcal{O}_k)$.
- Instead, if we remove one point and we consider the affine line: $\mathbb{A}^1(\mathbb{C}) = \{x \in \mathbb{C}\}$, then the rational points over k are $\mathbb{A}^1(k) = \{x \in k\}$ and the integral points are $\mathbb{A}^1(\mathcal{O}_k) = \{x \in \mathcal{O}_k\}$ and these two sets do not coincide. How can we recognize integral points between rational ones? If $k = \mathbb{Q}$, so $\mathcal{O}_k = \mathbb{Z}$, after embedding the affine line in the projective line, we can write every rational point as $\left[1 : \frac{a}{b}\right] = [b : a]$, $a, b \in \mathbb{Z}$, $\gcd(a, b) = 1$ and a point is integral if and only if there are no primes dividing b or, in other words, if and only if $[b : a] \not\equiv_p [0 : 1] = \infty$, which is the point we remove from the projective line to obtain the affine line, for every prime p.

For the discussion on general varieties we need a notion of reduction modulo a prime, so we need to introduce the concept of models.

Definition 1.1.2. Let X be a (quasi-)projective variety over a number field k with ring of integers $\mathcal{O}_k$. A **model of** X **over** k is a variety $\mathcal{X}$ with a dominant, flat morphism of finite type

$$\mathcal{X} \to \operatorname{Spec}(\mathcal{O}_k)$$

such that the fiber over the generic point η, $\mathcal{X}_\eta$ is isomorphic to X.
Let $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_k)$ be a non-archimedean prime, and denote by $\mathbb{F}_\mathfrak{p}$ its residue field. Define then the **reduction modulo** $\mathfrak{p}$ of X as the fiber

$$\mathcal{X}_\mathfrak{p} := \mathcal{X} \times_{\mathcal{O}_k} \operatorname{Spec}(\mathbb{F}_\mathfrak{p}).$$

Now we have the right setting to define the reduction modulo primes of points. To do it we need to be able to lift rational points to points over $\operatorname{Spec}(\mathcal{O}_k)$ and to do it, we will use the valuatative criterion of properness.

Theorem 1.1.3. *Valuatative criterion of properness [H, ch.II, theorem 4.7]*
Let $f : X \to Y$ be a morphism of finite type, with X noetherian. Then, f is proper if

and only if, for any field K, any valuation ring R with quotient field K and any pair of morphisms $\mathrm{Spec}(K) \to X$*,* $\mathrm{Spec}(R) \to Y$ *forming a commutative diagram:*

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

there exists a unique morphism $\mathrm{Spec}(R) \to X$ *making the whole diagram commutative.*

Let X be a (quasi-)projective variety with a proper model $f : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_k)$ (for example, if $\mathcal{X}$ is projective over $\mathrm{Spec}(\mathcal{O}_k)$, then the model is proper). If $P \in X(k)$ is a rational point of X over k, then it is identified with a k-morphism $P : \mathrm{Spec}(k) \to X$. We can apply the valuatative criterion of properness 1.1.3 to obtain a point: $\tilde{P} : \mathrm{Spec}(\mathcal{O}_k) \to \mathcal{X}$. In fact, for every prime $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$, let $\mathcal{O}_{k,\mathfrak{p}}$ be the local ring at $\mathfrak{p}$ and consider the commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, "P"] \arrow[d] & X \cong \mathcal{X}_\eta \arrow[r, hook] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}_{k,\mathfrak{p}}) \arrow[rr] \arrow[urr, dashed, "\exists! P_\mathfrak{p}"] & & \mathrm{Spec}(\mathcal{O}_k) \end{tikzcd}$$

By the valuatative criterion of properness, we obtain a unique point $P_\mathfrak{p}$ for every prime. Then, define

$$\tilde{P} : \mathrm{Spec}(\mathcal{O}_k) \to \mathcal{X}; \quad \mathfrak{p} \mapsto P_\mathfrak{p}(\mathrm{Spec}(\mathcal{O}_{k,\mathfrak{p}})),\ (0) \mapsto P$$

Definition 1.1.4. With the notations above, the **reduction modulo** $\mathfrak{p}$ of a rational point P is the image of $P_\mathfrak{p}$ in the fiber $\mathcal{X}_\mathfrak{p}$, i.e. it is $P_\mathfrak{p}(\mathrm{Spec}(\mathcal{O}_{k,\mathfrak{p}})) \times_{\mathcal{O}_k} \mathrm{Spec}(\mathbb{F}_\mathfrak{p})$. By abuse of notation we denote it by $P_\mathfrak{p}$.

Example 1.1.5. If k is a number field with ring of integers $\mathcal{O}_k$, $X \subseteq \mathbb{P}^n_k$ is a projective variety, $P = [p_0 : ... : p_n] \in X(k)$, then we claim that we can find coordinates for this point such that all $p_i \in \mathcal{O}_k$ and for every prime $\mathfrak{p}$ of $\mathcal{O}_k$, there exists an index $\tilde{i}$ such that $p_{\tilde{i}} \notin \mathfrak{p}$, so the reduction modulo $\mathfrak{p}$ of P is just the point $[p_0 \bmod \mathfrak{p} ... : p_n \bmod \mathfrak{p}]$ which is a well-defined point of $\mathbb{P}^n_{\mathbb{F}_\mathfrak{p}}$. Indeed, we can first multiply each p_i by an integer so that all p_i lie in $\mathcal{O}_k$, then we can factorize each ideal $(p_i) = \prod_\mathfrak{p} \mathfrak{p}^{e_i^\mathfrak{p}}$. Note that in these factorizations we are using only finitely many primes $\mathfrak{p}$ of $\mathcal{O}_k$. Let $\mathfrak{q}$ be one of those, without loss of generality we can assume $e_0^\mathfrak{q} \le e_i^\mathfrak{q}$ $\forall i = 0, ..., n$. As the class group of a number field is finite, there exists f' such that $\mathfrak{q}^{f'} = (a)$ is principal, define $f := f' - e_0^\mathfrak{q}$ and take an element $\alpha \in \mathfrak{q}^f$. Then, $[p_0 : ... : p_n] = [\alpha p_0 : ... : \alpha p_n]$ and $\alpha p_i \in \mathfrak{q}^{f'}$ $\forall i$, so $\alpha p_i = a\pi_i, \exists \pi_i \in \mathcal{O}_k$. So, $P = [\pi_0 : ... : \pi_n]$ and $\pi_0 \notin \mathfrak{q}$. We can apply this process to all the finitely many primes involved in the factorization to obtain the claim.

Now, we can define what an integral point is: a point that can be reduced modulo almost every prime. Later, we define another notion of integral points, the orbifold integral points. The idea of this latter definition is to remove some codimension one subvarieties from a projective variety X, but allowing "sufficiently high intersection" of points with these subvarieties modulo a finite set of primes.

Definition 1.1.6. Let X be a projective variety over a number field k with ring of integers $\mathcal{O}_k$, S a finite set of non-archimedean places of k. Let $f : \mathcal{X} \to \operatorname{Spec}(\mathcal{O}_{k,S})$ be a proper model of X over $\operatorname{Spec}(\mathcal{O}_{k,S})$, where $\mathcal{O}_{k,S}$ is the localization of $\mathcal{O}_k$ at S. The **S-integral points** of X are morphisms $P : \operatorname{Spec}(\mathcal{O}_{k,S}) \to \mathcal{X}$, which are sections of f.

Definition 1.1.7. An **orbifold pair** consists of an irreducible, normal, projective variety X over a number field k together with an effective $\mathbb{Q}$-divisor, $\Delta = \sum_{j=0}^{d} c_j D_j$, where:

- D_j are irreducible, distinct prime (Weil) divisors on X,
- $c_j = 1 - \frac{1}{m_j} \in (0, 1]$, for $m_j \in \mathbb{Z}_{>1}$ (with the convention $m_j = \infty$ if $c_j = 1$).

The **support** of the divisor, $\operatorname{supp}(\Delta)$ is the union of the prime divisors D_j.

Definition 1.1.8. A Weil divisor $D = \sum_{j=0}^{d} D_j$ on a variety X is said to be **of simple normal crossing (SNC)** if D_j are smooth subvarieties for all j and, for every point P of X, a local equation for D is $x_1 \cdot ... \cdot x_r$, for independent local parameters $x_i \in \mathcal{O}_{X,P}$, the stalk of the structure sheaf at the point P, with $r \leq dimX$.
An orbifold pair (X, Δ) is **smooth** if X is smooth and the divisor corresponding to $\operatorname{supp}(\Delta)$ is of simple normal crossing.

Definition 1.1.9. Let (X, Δ) be an orbifold pair and let K_X be the canonical bundle on X. The divisor $K_X + \Delta$ is called **the canonical bundle of the pair**.

To define orbifold-integral points we need to be able to reduce also the divisor in the orbifold pair modulo primes, so we need to introduce the notion of model for the orbifold pair.

Definition 1.1.10. Let (X, Δ) be an orbifold pair over a number field k with ring of integers $\mathcal{O}_k$. A **model of the pair** is a proper model $\mathcal{X} \to \operatorname{Spec}(\mathcal{O}_k)$ of X, together with a model $\mathcal{D} \to \operatorname{Spec}(\mathcal{O}_k)$ of the divisor D corresponding to $\operatorname{supp}(\Delta)$ such that $\mathcal{D}$ is a Cartier divisor of $\mathcal{X}$.

Arithmetic intersection numbers

To define what an orbifold integral point is, we need to "control" the intersection of a point with the orbifold divisor Δ modulo primes.
Let $(X, \Delta = \sum_j (1 - \frac{1}{m_j}) D_j)$ be a smooth orbifold pair and let $(\mathcal{X}, \mathcal{D}) \to \operatorname{Spec}(\mathcal{O}_k)$ be a model of the pair. Let $S \subseteq \operatorname{Spec}(\mathcal{O}_k)$ be a finite set of primes containing all the finitely many primes $\mathfrak{p}$ for which the fiber $(\mathcal{X}_\mathfrak{p}, \Delta_\mathfrak{p})$ is not smooth. Instead of working with $\mathcal{O}_k$, we work with $\mathcal{O}_{k,S}$.
Let $P \in X(k)$ be a closed k-point of X, assume that, for every j, $P \notin D_j$. Let g_j be a

local equation in an open subset containing P defining the component $\mathcal{D}_j$ corresponding to D_j in $\mathcal{D}$. As $P \notin D_j$, $g_j(P) \neq 0$, so the reduction of $g_j(P)$ modulo primes does not vanish for all but finitely many primes.

Definition 1.1.11. Let $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_k)$ be a finite prime. The **arithmetic intersection number,** $(P, D_j)_\mathfrak{p}$ is the largest integer t such that $g_j(P) \in \mathfrak{p}^t$.

Remark.

(i) The quantity $g_j(P) \in \mathcal{O}_{k,S}$ and $(P, D_j)_\mathfrak{p} \geq 1$ only for the finitely many primes dividing $g_j(P)$.

(ii) The number $(P, D_j)_\mathfrak{p}$ is independent of the choice of the local equation. Indeed, if h_j is another equation for the subvariety, then $h_j = g_j\gamma_j$, with $\gamma_j \in \mathcal{O}_X^*$, invertible around the point P, so $\gamma_j(P) \in \mathcal{O}_{k,S}^*$ is a unit modulo primes not in S.

Definition 1.1.12. Let (X, Δ) be a smooth orbifold pair over a number field k, S a finite set of places containing the ones over which (X, Δ) has "bad reduction". A point $P \in X(k)$ is called (S, Δ)**-integral** (resp. **classically** (S, Δ)**-integral**) if:

(i) for every j, $P \notin D_j$;

(ii) for every $\mathfrak{p} \notin S$ s.t. $(P, D_j)_\mathfrak{p} \geq 1$, then $(P, D_j)_\mathfrak{p} \geq m_j$ (resp. $m_j | (P, D_j)_\mathfrak{p}$).
We denote the set of such points $(X, \Delta)(S, k)$ (resp.$(X, \Delta)^*(S, k)$).

Remark. Note that, when $\Delta = 0$, these points are exactly the rational points of the projective variety X, whereas when Δ is a $\mathbb{Z}$-divisor (all $m_j = \infty$) these points are the integral points for the quasi-projective variety $X \smallsetminus \mathrm{supp}(\Delta)$. The other cases vary from the projective to the quasi-projective variety.

Example 1.1.13. Some examples on $\mathbb{P}^1$

- Let $\Delta = (0) + (\infty)$. If $\frac{a}{b} \in \mathbb{P}^1(\mathbb{Q})$, with $\gcd(a, b) = 1$, then the point $\frac{a}{b}$ corresponds to the section $[b : a] \in \mathbb{P}^1_\mathbb{Z}$ and the reduction modulo a prime p is $[b \bmod (p) : a \bmod (p)] \in \mathbb{P}^1_{\mathbb{F}_p}$. The points (0) and (∞) correspond respectively to the sections $[1 : 0]$ and $[0 : 1]$. So,

$$\left(\frac{a}{b}, (0)\right)_p \geq 1 \iff p|a;$$
$$\left(\frac{a}{b}, (\infty)\right)_p \geq 1 \iff p|b.$$
$$\Rightarrow (\mathbb{P}^1, \Delta)(\mathbb{Q}) = \{\frac{a}{b}\, |\text{there is no prime } p \text{ such that } p|a \text{ or } p|b\} = \{\pm 1\}$$

- Let $\Delta = (0) + (\infty)$ and $S = \{p_1, ..., p_s\}$ a finite set of primes. In this setting we do not ask any condition for the primes in S, so:

$$(\mathbb{P}^1, \Delta)(\mathbb{Q}, S) = \{\frac{a}{b}\, |\forall p \notin S,\ p \not| a \text{ or } p \not| b\} = \text{set of units of } \mathbb{Z}[S^{-1}].$$

- Let $\Delta = (0) + (1) + (\infty)$ and $S = \{p_1, ..., p_s\}$ a finite set of primes. Then, to the two conditions above, we are adding also:

$$[b : a] \not\equiv_p [1 : 1] \Leftrightarrow a \not\equiv_p b \quad \forall p \notin S$$
$$\Leftrightarrow \forall p \notin S,\ p \not| a - b \Leftrightarrow c := a - b \in \text{set of units of } \mathbb{Z}[S^{-1}].$$

  So the set $(\mathbb{P}^1, \Delta)(\mathbb{Q}, S)$ corresponds to the solutions of the S-unit equation $a = b + c$.

- Let $\Delta = \left(1 - \frac{1}{s}\right)(0) + \left(1 - \frac{1}{r}\right)(1) + \left(1 - \frac{1}{q}\right)(\infty)$, $q, r, s \geq 2$. In this case, whenever an intersection number is strictly positive, then it has to be $\geq q, r, s$. So, $(\mathbb{P}^1, \Delta)(\mathbb{Q})$ corresponds to the solutions of the unit equation $a = b+c$, where a is an s-powerful integer, b is a q-powerful integer and c is an r-powerful integer (an integer x is called k-powerful if, for every prime p with $p|x$, in fact $p^k|x$).

Definition 1.1.14. An orbifold pair (X, Δ) is said to be **potentially dense** if there exists k'/k, a finite extension and S' a finite set of primes such that $(X, \Delta)(S', k')$ is Zariski dense in X. Conversely, it is said to be **mordellic** if there exists an open dense subset $U \subseteq X$ such that $(X \cap U, \Delta)(S', k')$ is finite for every finite extension k'/k and every finite set of primes S'.

Example 1.1.15. These definitions are not exclusive, there can be varieties that are neither mordellic, nor potentially dense. For example, if $X = F \times C$, with F a curve of genus ≤ 1 and C a curve of genus ≥ 2. Indeed, consider the second projection, p, C is mordellic and F is potentially dense (we will see this in the next section in theorem 1.2.1), thus k-rational points of X are concentrated in the finitely many fibers of p over the rational points of C and these fibers are potentially dense.

1.2 Orbifold integral points on curves

In this section we state what is known so far about finiteness of orbifold integral points on curves and the orbifold Mordell conjecture for curves. We discuss some of the proofs in the next sections. When the divisor $\Delta = 0$ the behavior of rational points on a curve is completely determined by the genus and so by the degree of the canonical divisor of the curve (Faltings' theorem). In the case where the orbifold divisor Δ coincides with its support (quasi-projective case) the degree of the canonical divisor plus the degree of Δ determines if the orbifold pair is mordellic or potentially dense (Siegel's theorem). It is proven that also for the classical (S, Δ)-integral points this is the right invariant to look at and it is conjectured that the same conclusion holds in the non-classical setting as well.

Theorem 1.2.1. *Let C be a connected, smooth, projective curve over a number field k, denote by g its genus and by K_C its canonical divisor. Then:*

- *if $g = 0$ (deg(K_C)$= -2 < 0$), there exists a quadratic extension k'/k such that $C(k') \neq \emptyset$. Then, C over k' is isomorphic to $\mathbb{P}^1_{k'}$, so it has infinitely many rational points;*

- *if $g = 1$ (deg(K_C)$= 0$), after at most a finite extension of the base field k'/k, $C(k')$ is an elliptic curve with infinitely many rational points;*

- *if $g \geq 2$ (deg(K_C)> 0), then $C(k')$ is finite for every finite extension k'/k of the base field. (Faltings' theorem)*

Theorem 1.2.2. *Let C be a connected, smooth, projective curve over a number field k and $D = \sum_{j=0}^{s} P_j$ a smooth divisor on it, where P_j are distinct points on C. Denote by K_C its canonical divisor. Also, consider S to be a finite set of places of k. Then:*

- *if deg$(K_C + D) \leq 0$, after at most a finite extension of the base field k'/k, the set of (S, D)-integral points is infinite;*

- *if deg($K_C + D$)> 0, then the set of (S, D)-integral points is finite for every finite extension k'/k of the base field. (Siegel's theorem)*

Remark. The first part of this theorem is straight forward. Indeed, $\deg(D) > 0$ and $\deg(K_C + D) \leq 0$ if and only if $C' := C \setminus D$ is one of the followings:

- C' is the affine line (projective line minus one point);

- C' is the formal multiplicative group $\mathbb{G}_m$ (projective line minus two points, so isomorphic to a plane affine curve with equation $xy = 1$).

In both cases the S-integral points are clearly Zariski dense (possibly after a finite extension of the base field).

The general case is still a conjecture: orbifold Mordell conjecture for curves.

Conjecture 1.2.3. *Let (C, Δ) be a curve orbifold pair over a number field k, denote by K_C the canonical divisor. Let S be a finite set of places of k. Then $(C, \Delta)(k', S')$ (resp. $(C, \Delta)^*(k', S')$) is finite for every finite extension k'/k and S' finite set of places of k' if and only if deg$(K_C + \Delta) > 0$.*

Remark. Results we have so far.

- The conjecture for classical integral point is completely solved and we will talk about it in section 1.4.

- Note that $(C, \Delta)^*(k, S) \subseteq (C, \Delta)(k, S)$, so, if $\deg(K_C + \Delta) \leq 0$, then $(C, \Delta)^*(k, S)$ is infinite and so must be $(C, \Delta)(k, S)$.

- Moreover, if the genus of C is ≥ 2, then $(C, \Delta)(k, S) \subseteq (C, 0)(k, S)$ and the latter is finite by Faltings' theorem and consequently the former is finite as well.

- If Δ, Δ' are two orbifold divisor such that $\Delta' > \Delta$ (which means that $\Delta' - \Delta$ is effective), then $(C, \Delta')(k, S) \subseteq (C, \Delta)(k, S)$, hence if the second is finite, so is the first.

- By these remarks we have reduced to study only some "minimal" cases with $\deg(K_C + \Delta) > 0$:

  (i) if the curve has genus 1, so the degree of the canonical divisor is 0, then the minimal case is with $\Delta = \left(1 - \frac{1}{m}\right)(a)$ for a closed point a of the curve.

  (ii) if the curve has genus 0 the canonical divisor has degree -2, so the degree of Δ has to be strictly bigger than 2. Call s the number of points in the support of Δ, so $\Delta = \sum_{j=1}^{s}\left(1 - \frac{1}{m_j}\right)(a_j)$ for closed points a_j. Then $\sum_{j=1}^{s}\left(1 - \frac{1}{m_j}\right) > 2$, which implies $\sum_{j=1}^{s}\left(\frac{1}{m_j}\right) < s - 2$.
  So the minimal cases for a curve with genus 0 are:

$$
\begin{aligned}
s = 3 &\quad (m_1, m_2, m_3) \in \{(2,3,7), (2,4,5), (3,3,4)\}; \\
s = 4 &\quad (m_1, m_2, m_3, m_4) = (2,2,2,3) \\
s = 5 &\quad (m_1, m_2, m_3, m_4, m_5) = (2,2,2,2,2)
\end{aligned}
$$

1.3 Siegel's theorem

We do not discuss in this thesis the proof of Faltings' theorem, but we see the one of Siegel's theorem, using the approach of P. Corvaja and U. Zannier, based on the subspace theorem.

Let k be a number field, for every valuation ν, on $\mathbb{A}_k^n$, we can define a norm. If $P = (x_1, ..., x_n) \in \mathbb{A}_k^n$, denote by $|\cdot|_\nu$ the corresponding norm on k and let:

$$\|P\|_\nu := \max_{i=1,\dots,n}\{|x_i|_\nu\}$$

Look at $\mathbb{A}_k^n$ as a subset of $\mathbb{P}_k^n$ and define, for the points in $\mathbb{P}_k^n \setminus \mathbb{A}_k^n$, the points at infinity, $\|P\|_\nu := \infty$. We need to define also another element, the **height** of a point:

$$H(P) := \prod_{\nu} \|P\|_\nu,$$

where the product is taken over all the valuations on k.

Theorem 1.3.1. *Subspace theorem [7, Subspace theorem], equivalent to [8, theorem 1F] Let S be a finite set of non-archimedean places of k, T the union of S and the set of the arcimedean ones and $N \geq 2$ an integer. Consider, for all $\nu \in T$, N linearly independent linear forms in N variables with coefficients in k_ν, the completion of k with respect to ν: $L_{\nu,1}(X_1, ..., X_N), ..., L_{\nu,N}(X_1, ..., X_N)$. Then, the solutions $(x_1, ..., x_N) \in \mathcal{O}_{k,S}^N$ to the inequality*

$$\prod_{\nu \in T} \prod_{i=1}^{N} |L_{\nu,i}(x_1, ..., x_N)|_\nu < H(x_1, ..., x_n)^{-\epsilon}$$

lie in the union of finitely many proper linear subspaces of $\mathbb{A}^n_k$.

We need also another classical result, Chevalley–Weil theorem, which is true also for higher dimensional varieties.

Theorem 1.3.2. *Chevalley–Weil*
Let $f : X \to Y$ *be a finite étale covering of normal projective varieties defined over a number field* k. *Then, there exists an integer* $T > 0$ *such that for all* $P \in Y(k)$, *there exists* $P' \in X(k')$ *such that* $f(P') = P$ *and the relative discriminant of* k'/k *divides* T. *In particular, applying Hemite–Minkowski theorem 1.4.3, we find* k'/k *finite extension such that* $Y(k) \subseteq f(X(k'))$.

Theorem 1.3.3. *Siegel*
Let $C \subseteq \mathbb{A}^n_k$ *be an affine curve over a number field* k *and* $\widetilde{C} \subseteq \mathbb{P}^n_k$ *its projective closure. Let* d *be the number of points at infinity of the curve, i.e.* $d := \#\widetilde{C} \setminus \mathbb{A}^n_k$ *and let* D *be the divisor which is the sum of the points at infinity of the curve. If deg(*K_C*)+*$d > 0$, *then the set of* (S, D)*-integral points is finite for every finite extension* k'/k *of the base field and finite set of places* S.

Remark. This version is actually equivalent to the one stated in theorem 1.2.2. Indeed, for every curve C and every divisor which is a sum of points, $D = \sum_{i=1}^{d} P_i$, we can find an embedding in a projective space $\mathbb{P}^N$, with $N \gg 0$, such that the points at infinity of the curve are exactly $P_1, ..., P_d$.

Proof. First of all, note that we can suppose that the curve is smooth. Indeed, if not, we can prove the result for its normalization and conclude for the curve itself.

Step 1 We can reduce to prove that the number of (S, D)-integral points is finite whenever $d \geq 3$. Indeed, if $d = 1$ or 2, then, for deg(K_C)+d to be > 0, the genus of the curve must be positive. For any smooth curve with positive genus g there are unramified covers of any degree; they can be constructed with standard techniques of covering spaces as the first homology group of the complex Riemann surface associated with the curve is a free abelian group of rank $2g$. Choose one of those with degree ≥ 3: $f : \widetilde{C}' \to \widetilde{C}$ and set $D' := f^{-1}(D)$, the support of D' contains more than 3 points, thus, applying the remark above, we get the result for the curve $\widetilde{C}'$ with the divisor D'. Using Chevalley–Weil theorem 1.3.2, we can conclude the same also for the original curve as there exists k'/k finite extension such that the (S, D)-integral points of $\widetilde{C}$ over k are contained in the image of the ones of $\widetilde{C}'$ over k'.

Step 2 The statement we are left to prove now is therefore the following.
Let $C \subseteq \mathbb{A}^n_k$ *be a smooth affine curve over a number field* k *with* $\widetilde{C} \subseteq \mathbb{P}^n_k$ *its projective closure, smooth as well. Denote by* D *the divisor corresponding to the points at infinity. If the support of* D *has degree at least three, then the set of* (S, D)*-integral points is finite for every finite extension* k'/k *of the base field and finite set of places* S.
Let us suppose that the conclusion of the theorem was not true, i.e. there exists

an infinite sequence of distinct (S, D)-integral points $\{P_1, P_2, ...\}$. Let S' be the union of S and the set of archimedean places, for every $\nu \in S'$, consider k_ν, the completion of k with respect to ν. The points P_i can be considered inside $\widetilde{C}(k_\nu)$, after a base change of $\widetilde{C}$ and this latter curve is compact with respect to the norm induced by ν on the projective space $\mathbb{P}^n_k$. Hence we can extract a converging subsequence. As S' is a finite set, passing to subsequences, we can suppose that $(P_i)_{i\in\mathbb{N}}$ converges for all $\nu \in S'$; call R_ν the ν-adic limits.
Let $S^* := \{\nu \in S' | R_\nu \in \mathrm{Supp}(D)\}$, note that this set is not empty. Indeed, if it was, then, up to finitely many points, $\|P_i\|_\nu$ would be bounded for all $\nu \in S'$ and this would imply that the height of P_i was bounded as well, as the coordinates of these points are in $\mathcal{O}_{k,S}$. This leads to a contradiction because there are only finitely many points with bounded height.
If $\mathscr{F}$ is an $\mathcal{O}_{\widetilde{C}}$-module associated with a divisor E (for the correspondence, see theorem 2.1.2), denote by $h^0(\widetilde{C}, \mathscr{F}) = h^0(\widetilde{C}, E)$ the dimension over k of the set of global sections of $\mathscr{F}$, denoted by $H^0(\widetilde{C}, \mathscr{F}) = H^0(\widetilde{C}, E)$. By Riemann–Roch teorem [H, ch.IV, theorem 1.3], for $N \gg 0$, $h^0(\widetilde{C}, K_{\widetilde{C}} - ND) = 0$ and $h^0(\widetilde{C}, ND) = Nd + 1 - g := M + 1$, where g is the genus of the curve. Let $V_N := H^0(\widetilde{C}, ND)$ and $\mathcal{B} = \{f_0, ..., f_M\}$ a basis of it over k. After multiplying by a suitable nonzero constant, we may assume that $f_j(P_i) \in \mathcal{O}_{k,S}\ \forall i, j$.
For every $\nu \in S^*$, consider a filtration $V_N = W_{\nu,1} \supseteq W_{\nu,2} \supseteq ...$ defined as:

$$W_{\nu,j} := \{f \in V_N | \mathrm{ord}_{R_\nu} f \geq j - N - 1\} = H^0(\widetilde{C}, ND - NR_\nu + (j - N - 1)R_\nu)$$

where $\mathrm{ord}_{R_\nu} f$ is the order of vanishing at R_ν of the function f.
As we supposed that the degree of D is ≥ 3, $ND - NR_\nu + (j - N - 1)R_\nu$ is effective. Applying Riemann–Roch theorem again, for $N \gg 0$, we get $h^0(\widetilde{C}, ND - NR_\nu + (j - N - 1)R_\nu) = j - N - 1 + Nd - N + 1 - g$, whence $\dim(W_{\nu,j}/W_{\nu,j+1}) \leq 1$ and $\dim W_{\nu,j} \geq M - j + 1$.
For every $\nu \in S^*$, choose a basis $\mathcal{B}_\nu$ of V_N that contains a basis of each $W_{\nu,j}$ (note that, after a finite number of steps, $W_{\nu,j} = 0$). Each element of this new basis can be expressed as a linear combination $L_{\nu,j}(f_0, ..., f_M)$ of elements in $\mathcal{B}$ with coefficients in the algebraic closure of k. Moreover,

$$\mathrm{ord}_{R_\nu} L_{\nu,j} \geq j - N - 1$$

Let t_ν be a local parameter at R_ν, then $L_{\nu,j}(f_0, ..., f_M) = t_\nu^{j-N-1} F$, where F is a regular function. Since $\{P_i\}_{i\in\mathbb{N}}$ tends to R_ν, in the ν-adic norm $F(P_i)$ is eventually bounded and

$$|L_{\nu,j}(f_0(P_i), ..., f_M(P_i))|_\nu \leq \alpha |t_\nu(P_i)|_\nu^{j-N-1}$$

for a constant α. Furthermore:

$$\mathrm{ord}_{R_\nu} \prod_{j=1}^{M+1} L_{\nu,j}(f_0, ..., f_M) \geq \sum_{j=0}^{M}(-N+j) = \frac{M}{2}((d-2)N + O(1)) > (d-2)\frac{N^2}{2} + O(N)$$

and this quantity is strictly positive.
For all $\nu \in S' \setminus S^*$, let $L_{\nu,j} := f_{j-1}$.
Let $\mathbf{x} := (f_0(P_i), ..., f_M(P_i))$. If $\nu \in S' \setminus S^*$, $\|\mathbf{x}\|_\nu$ is uniformly bounded for all but finitely many P_i's as the sequence has a finite limit. Also if $\nu \notin S'$, $\|\mathbf{x}\|_\nu$ is uniformly bounded since the P_i's are (S, D)-integral points. Therefore, we can conclude that:

$$\prod_{\nu \in S'} \prod_{j=1}^{M+1} |L_{\nu,j}(\mathbf{x})|_\nu \leq \beta \prod_{\nu \in S^*} |L_{\nu,j}(\mathbf{x})|_\nu \leq \gamma \prod_{\nu \in S^*} |t_\nu(P_i)|_\nu^{(d-2)N^2+O(N)}$$

for some constants β and γ. As the order of f_j at each R_ν is at least $-N$, we can bound in a similar way the height of $\mathbf{x}$:

$$H(\mathbf{x}) \leq \delta \prod_{\nu \in S^*} \max |f_j(P_i)|_\nu \leq \epsilon \prod_{\nu \in S^*} |t_\nu(P_i)|^{-N}$$

for some constants δ and ϵ. From these last two inequalities, we conclude that:

$$\prod_{\nu \in S'} \prod_{j=1}^{M+1} |L_{\nu,j}(\mathbf{x})|_\nu \leq \zeta H(\mathbf{x})^{(2-d)(N/2)+O(1)}$$

for a constant ζ. As $H(\mathbf{x})$ goes to infinity with i (as otherwise there would be only finitely many points in the sequence), we can choose μ such that, for all but finitely many P_i's,

$$\prod_{\nu \in S'} \prod_{j=1}^{M+1} |L_{\nu,j}(\mathbf{x})|_\nu \leq H(\mathbf{x})^{-\mu}.$$

Hence, we can apply the subspace theorem 1.3.1 which tells us that the solution points of the inequality lie in finitely many hyperplanes. In particular, there exists an hyperplane that contains infinitely many of the points $(f_0(P_i), ..., f_M(P_i))$. However, the functions f_j are linearly independent, thus this situation cannot happen.

qed

1.4 Classical orbifold Mordell conjecture

1.4.1 Tools

The next goal is to see the proof of the calssical version of Mordell orbifold conjecture. The strategy is to reduce to use the finiteness results we already have by means of suitable ramified covers $\pi : C' \to C$. The first part of this section gives the relation between rational points on C' and orbifold-integral points on C.

Before starting the discussion, we recall some results from number theory which are used later.

Theorem 1.4.1. *Purity of branch loci [N, theorem 41.1]*
Let $(R, \mathfrak{m})$ be a regular local ring and $(P, \mathfrak{n})$ a normal local ring which dominates R and which is a ring of quotients of a finite separable integral extension of R. If every height one prime in P is unramified over R, then P is unramified over R.

Proposition 1.4.2. *Let l/k be a finite extension of number fields, then a prime $\mathfrak{p}$ of k is ramified in l if and only if it divides the relative discriminant of the extension. Moreover, the biggest power e of the prime dividing the relative discriminant is bounded by a function which depends only on the degree $[l : k]$ of the extension (not on the field l and neither on the chosen prime $\mathfrak{p}$).*

Theorem 1.4.3. *Hermite-Minkowski*
For every $N \in \mathbb{N}$, there are only finitely many number fields $k/\mathbb{Q}$ such that $|disc(k/\mathbb{Q})| \leq N$.

Definition 1.4.4. Let $\pi : C \to C'$ be a surjective (hence finite) regular map over a number field k of smooth projective curves with a "good model" over $\mathrm{Spec}(\mathcal{O}_{k,S})$, with S a finite set of non-archimedean places of k. Let $\Delta = \sum_j \left(1 - \frac{1}{m_j}\right) P_j$ be an orbifold smooth divisor on C. We say π is **classically orbifold étale over** Δ if it is unramified outside $\mathrm{supp}(\Delta)$ and if, for every j and every $a \in \pi^{-1}(P_j)$, the ramification order at a is $e_a = m_j$.

Remark. Asking that π is classically orbifold étale over Δ is equivalent to asking $\pi^*(K_C + \Delta) = K_{C'}$, where K_C and $K_{C'}$ are the canonical divisors of the two curves (Riemann–Hurwitz' formula).

Definition 1.4.5. Given a ramified cover $\pi : C' \to C$, we can define the group of **deck transformations** associated with it, $\mathrm{deck}(\pi) = \{f : C' \to C' | \pi(f(a)) = \pi(a)\ \forall a \in C'\}$. A cover π is said to be **Galois** if its group of deck transformations acts transitively on the fibers, $\#\mathrm{deck}(\pi) = \deg(\pi)$ and the image of the fundamental group of C' under the map induced by π, is a normal subgroup of the fundamental group of C.

Consider a smooth orbifold pair $(C, \Delta = \sum_j \left(1 - \frac{1}{m_j}\right) P_j)$ and a Galois, classically orbifold étale cover over Δ, $\pi : C' \to C$ with a model over $\mathrm{Spec}(\mathcal{O}_{k,S})$ for S a finite set of non-archimedean places of k. This cover induces a finite Galois [1] extension on the function fields of the two curves with Galois group G: $\pi^* : k(C) \to k(C')$. So $k(C') \simeq \frac{k(C)[x]}{(p(x))}$, there exists a polynomial $p(x) \in k(C)[x]$. Take a rational closed point of C, $a \in C(k) \setminus \mathrm{supp}(\Delta)$ and consider the specialization of the field extension at a. Choosing coordinates we can have a more explicit description of the situation. Locally in an affine open subset around a, $k(C)$ can be described as $\mathrm{Quot}(\frac{k[y_1,...,y_n]}{I(C)})$, where $I(C)$ is the ideal associated with the curve in this affine and $p(x) = p(y_1, ..., y_n, x)$. With these

[1] the group of deck transformations corresponds to this Galois group, so, as the former acts transitively on the fibers by definition, the second acts transitively on the roots of the polynomial defining the extension, whence the fact that the extension is Galois.

coordinates, $a = (a_1, ..., a_n)$ and the specialization at a of the extension can be described as:

$$k(C)_{|a} \simeq \frac{k[y_1, ..., y_n]}{(I(C), y_1 - a_1, ..., y_n - a_n)} \simeq k \to k(C')_{|a} \simeq \frac{k[x]}{(p(a_1, ..., a_n, x))} \simeq l_1 \oplus ... \oplus l_t$$

for some $t \in \mathbb{N}$ and some fields l_i's, which correspond to the irreducible factors p_i's of $p(a_1, ..., a_n, x) = \prod_{i=1}^t p_i(a_1, ..., a_n, x)$. Let l_a be the compositum of all l_i's. Note that there exists a bound D for the degree of the extension $[l_a : k]$ which does not depend on the particular point chosen (for instance we may take $D = d^d$, where $d = \deg(p(x))$). Moreover, each field l_i is Galois over k as in $k(C')_{|a}$ there must be all the roots of $p(a_1, ..., a_n, x)$ because $k(C')$ is the splitting field of $p(x)$, so each l_i must contain all the roots of its corresponding irreducible factor p_i. Hence, l_a is Galois over k.

Remark. The map π induces an inclusion also of $\mathcal{O}_k[C]$ (resp. $k[C]$) in $\mathcal{O}_k[C']$ (resp. $k[C']$). Besides, $\mathcal{O}_k[C]$ (resp. $k[C]$) is locally defined as $\frac{\mathcal{O}_k[y_1,...,y_n]}{I(C)}$ (resp. $\frac{k[y_1,...,y_n]}{I(C)}$), where we can assume that $I(C)$ admits a set of generators of polynomials with coefficients in $\mathcal{O}_k$, and $\mathcal{O}_k[C']$ (resp. $k[C']$) denotes the integral closure of $\mathcal{O}_k[C]$ (resp. $k[C]$) in $k(C')$.

Definition 1.4.6. A model of the cover π over $\mathrm{Spec}(\mathcal{O}_{k,S})$ is called a **good model** if the primes of $\mathcal{O}_k$ (viewed as primes in $\mathcal{O}_k[C]$) which ramify in $\mathcal{O}_k[C']$ are contained in S_{bad}, the set of primes dividing the order of G and the primes where two points in the support of Δ meet.

Proposition 1.4.7. *In the above setting, suppose also that π admits a good model. If $a \in C(k) \setminus \mathrm{supp}(\Delta)$, then:*

(i) the finite primes of k which ramify in l_a are contained in the set $\tilde{S} = S_{bad} \cup S_a$, where S_{bad} is the set of primes dividing the order of G and the primes where two points in the support of Δ meet and S_a is the set of primes $\mathfrak{p}$ such that $(a, P_j)_{\mathfrak{p}} > 1$ for a point P_j in the support of Δ;

(ii) if for every $\mathfrak{p} \in S_a$, $(a, P_j)_{\mathfrak{p}} \equiv_{m_j} 0$, then l_a is unramified at $\mathfrak{p}$.

Proof. (i) Let $\mathfrak{p} \notin S$ and denote by $\mathcal{O}_{\mathfrak{p}}$ the local ring of integers of $k_{\mathfrak{p}}$, the completion of k with respect to $\mathfrak{p}$. Let ξ be a generator of the maximal ideal of $\mathcal{O}_{\mathfrak{p}}$ and t_a a uniformizer at a for the curve C with the scalars extended to $k_{\mathfrak{p}}$. Consider (t_a) as an ideal of $\mathcal{O}_{\mathfrak{p}}[t_a]$. The map π defines a finite extension of the field $k_{\mathfrak{p}}(t_a)$.

In coordinates, keeping the notations used above, this extension is defined by the image of the polynomial $p(y_1, ..., y_n, x)$ in $k(t_a)[x]$ by the map sending y_i to its image in the local ring (identified with $k(t_a)$). Denote this resulting polynomial as $p(t_a, x)$. The extension defined by π corresponds to the extension $\frac{k(t_a)[x]}{(p(t_a,x))} = \prod_i L_i$, for some fields L_i. By switching the affine patch we are choosing, if necessary, we can assume that $\mathrm{ord}_a(y_i) \geq 0$, so that we can consider also the extension defined by π on the rings of integers. Denote by $\mathcal{O}_{\mathfrak{p}}[C']$ the integral closure of $\mathcal{O}_{\mathfrak{p}}[t_a]$

in $\frac{k(t_a)[x]}{(p(t_a,x))}$. Choose a prime ideal $\mathscr{A}$ in $\mathcal{O}_{\mathfrak{p}}[C']$ containing t_a, note that this choice corresponds to the choice of an irreducible factor $p_i(t_a, x)$ of $p(t_a, x)$. Remark that:

$$\frac{\mathcal{O}_{\mathfrak{p}}[C']_{\mathscr{A}}}{\mathscr{A}\mathcal{O}_{\mathfrak{p}}[C']_{\mathscr{A}}} \simeq \left(\frac{k(t_a)[x]}{p_i(t_a,x)}|\text{specialized at } a\right)_{\mathfrak{q}} \simeq (l_i)_{\mathfrak{q}}$$

where $\mathfrak{q}$ is a place above $\mathfrak{p}$ [2]. Let $\mathfrak{m} = (\xi, t_a)$ be the maximal ideal of $\mathcal{O}_{\mathfrak{p}}[t_a]$ and $\mathfrak{n}$ a maximal ideal in $\mathcal{O}_{\mathfrak{p}}[C']$ over $\mathfrak{m}$ and containing $\mathscr{A}$. The height one primes contained in $\mathfrak{m}$ are only (ξ) and (t_a) and we claim that they are both unramified in $\mathcal{O}_{\mathfrak{p}}[C']$. Indeed, the only finite Galois extensions of $k_{\mathfrak{p}}(t_a)$ are of the form $l_{\mathfrak{q}}(t_a^{1/e})$, where e corresponds to the ramification of π at a. But by hypothesis $(a, P_j)_{\mathfrak{p}} = 0$ for every j, so $e = 1$, hence (t_a) is unramified. On the other hand, (ξ) is unramified because $l_{\mathfrak{q}}$ is unramified over $k_{\mathfrak{p}}$ as $\mathfrak{p} \notin S_{\text{bad}}$. Thus, we are in the situation of theorem of purity of branch loci 1.4.1, so, applying this result we conclude that $\mathfrak{m}$ is unramified in $\mathcal{O}_{\mathfrak{p}}[C']$, which is equivalent to saying that its maximal ideal is generated by ξ and t_a. So, specializing at a, we get that $\left(\mathcal{O}_{\mathfrak{p}}[C']_{|\text{specialized at } a}\right)_{\mathfrak{n}} = (l_i)_{\mathfrak{q}}$ has maximal ideal generated by ξ, i.e. $\mathfrak{p}$ is unramified in l_i and this is true for all i, so it holds also for the compositum l_a.

(ii) Let j be the index such that $(a, P_j)_{\mathfrak{p}} > 0$ for a prime $\mathfrak{p} \notin S_{\text{bad}}$ and let t_j be a uniformizer for $P_j \in C/k$. The map π induces a finite field extension:

$$k(t_j) \to \frac{k(t_j)[x]}{p_i(t_j,x)} = L_i$$

where $p_i(t_j, x)$ is an irreducible factor of $p(t_j, x)$, polynomial defined as in the previous part.

Let $\mathscr{A}$ be a place of $k(t_j)$ containing both $\mathfrak{p}$ and t_j and $\bar{\mathscr{A}}$ a place of L_i above it. Complete the fields with respect to these two ideals to get an inclusion of Puiseux series:

$$k_{\mathfrak{p}}((t_j)) \hookrightarrow l_{\mathfrak{q}}((t_j^{1/m_j}))$$

where $l_{\mathfrak{q}}$ is an unramified extension of $k_{\mathfrak{p}}$ as $\mathfrak{p} \notin S_{\text{bad}}$. The induced extension has this form, assuming that all $m_j{}^{\text{th}}$ roots of unity are in $k_{\mathfrak{p}}$ because all finite Galois extensions of the Puiseux series field are in this form and locally the map π can be described as $t_j \mapsto t_j^{m_j}$. As we are considering a point a for which $(a, P_j)_{\mathfrak{p}} > 0$, it makes sense to evaluate Puiseux series at a (they converge). So, specializing at a we get that:

$$(l_i)_{\mathfrak{p}} = l_{\mathfrak{q}}((t_j(a)^{1/m_j})).$$

But, by assumption, $(a, P_j)_{\mathfrak{p}} = m_j e$ for a positive integer e, thus the valuation of $t_j(a)^{1/m_j}$ at $\mathfrak{q}$ is an integer, i.e. the valuation of $(l_i)_{\mathfrak{p}}$ is contained in $\mathbb{Z}$, which in turn implies that the extension is unramified.

qed

[2] The irreducible factors of $p(a, x)$ correspond to the irreducible factors of $p(t_a, x)$ by Hensel's lemma.

Remark. This theorem cannot be used for the non-classical conjecture as the proof of it requires the divisibility of the arithmetic intersection numbers by the multiplicities m_j.

The next result allows us to compare orbifold-integral points on C and rational points on C', when we have a "good" cover. It is an orbifold version of Chevalley–Weil theorem 1.3.2 for curves.

Theorem 1.4.8. *Let $\pi : C' \to C$ be a classically orbifold étale cover over $\Delta = \sum_j \left(1 - \frac{1}{m_j}\right) P_j$ with a good model over $\operatorname{Spec}(\mathcal{O}_{k,S})$. Then:*

(1)
$$\pi(C'(k) \setminus \{\textit{ramification points}\}) \subseteq (C, \Delta)^*(k, S);$$

(2) there exists k'/k, finite extension, such that
$$\pi(C'(k')) \supseteq (C, \Delta)^*(k, S).$$

Proof. (1) Let $a = \pi(b) \in C(k)$ for a non-ramification point $b \in C'(k)$. If there exist j and $\mathfrak{p} \notin S$ such that $(a, P_j)_{\mathfrak{p}} \geq 1$, then by hypothesis we know that the cover induces a map $\pi_{\mathfrak{p}}$, over the residue field $\mathbb{F}_{\mathfrak{p}}$, which is ramified at P_j with order m_j. But $a \equiv_{\mathfrak{p}} P_j$, so if we call $t_j^{\mathfrak{p}}$ and $t_b^{\mathfrak{p}}$ the uniformizers of C and C' over $\mathbb{F}_{\mathfrak{p}}$ at respectively $a \equiv P_j$ and b, then the map induced on the local rings looks like:

$$\mathcal{O}_{\mathfrak{p},a} \to \mathcal{O}'_{\mathfrak{p},b}; \quad t_j^{\mathfrak{p}} \mapsto t_b^{\mathfrak{p} m_j}$$

i.e. $\pi_{\mathfrak{p}}^*(t_j^{\mathfrak{p}}) = t_b^{\mathfrak{p} m_j}$. Choose t_j and t_b lifts of $t_j^{\mathfrak{p}}$ and $t_b^{\mathfrak{p}}$ over $\mathcal{O}_{k,S}$ with $\operatorname{ord}_{P_j}(t_j) = 1$, so the equation defining P_j locally looks like $t_j = 0$. Then, $\pi^*(t_j) \equiv_{\mathfrak{p}} t_b^{m_j}$, so $t_j(a) = t_j(\pi(b)) \equiv_{\mathfrak{p}} t_b(b)^{m_j}$. By assumption, $t_j(a) \equiv_{\mathfrak{p}} 0 \equiv_{\mathfrak{p}} t_b^{m_j}$, hence, as $\mathfrak{p}$ is prime, $t_b \in \mathfrak{p}$. Let e be the largest integer such that $t_b \in \mathfrak{p}^e$, than $m_j e$ is the largest integer such that $t_j(a) \in \mathfrak{p}^{m_j e}$, which means $(a, P_j)_{\mathfrak{p}} = e m_j$, so $a \in (C, \Delta)^*(k, S)$.

(2) Let $a \in (C, \Delta)^*(k, S)$, by enlarging S if necessary, we can assume $S_{\text{bad}} \subseteq S$. For every j and $\mathfrak{p} \notin S$, by definition of classically integral points, either $(a, P_j)_{\mathfrak{p}} = 0$ or $m_j | (a, P_j)_{\mathfrak{p}}$. So, by proposition 1.4.7, $\mathfrak{p}$ is unramified in l_a. Moreover, we know there is a bound D for the degree of l_a over k which is independent of a: $[l_a : k] \leq D$. So, by proposition 1.4.2, the relative discriminant of $[l_a : k]$ is bounded by a product of primes in S, raised to a bounded power, which in turns imply that the discriminant of l_a over k is bounded. Then, by Hermite-Minkowski's theorem 1.4.3, we can conclude that there are only finitely many possibilities for these fields l_a. Let k' be the compositum of all of them, this is the finite extension of k we were looking for.

qed

In the second part of this section we discuss Riemann existence theorem, which is used to produce classically orbifold étale covers in some of the cases we will deal with in the next section. This theorem has two versions, one algebraic and one analytic and the tool we will use is actually the theorem that allows to switch from the algebraic to the analytic version. By the sake of completeness, we state all the results here, but we prove only the theorem which is needed in the discussion to follow.

Consider the function field of $\mathbb{P}^1_{\mathbb{C}}$, namely $\mathbb{C}(x)$, and let L be a finite Galois extension of it with Galois group G. Let $p \in \mathbb{P}^1(\mathbb{C})$ be a closed point and

$$\theta_p : \mathbb{C}(x) \to \mathbb{C}((t)); \quad x \mapsto t + p$$

(so that t can be considered a uniformizer at p). All Galois extensions of $\mathbb{C}((t))$ are of the form $\mathbb{C}((t^{1/e}))$ with Galois group generated by the element $\omega_e : t^{1/e} \mapsto t^{1/e}\zeta_e$ (ζ_e a primitive e^{th} root of unity). The map θ_p extends (uniquely up to conjugation) to $\psi_p : L \to \mathbb{C}((t^{1/e}))$. Define g_p to be the restriction of ω_e to L with respect to the map ψ_p, so $g_p \in G$, it has order e and is unique up to conjugation (the conjugation corresponds to the choice of an ideal above (t)). So, the extension L together with the point p determines a conjugacy class of G, namely C_p, whose elements have all order e. This number e is called the **ramification index of L at p** and the point p is said to be **ramified** when $e > 1$.

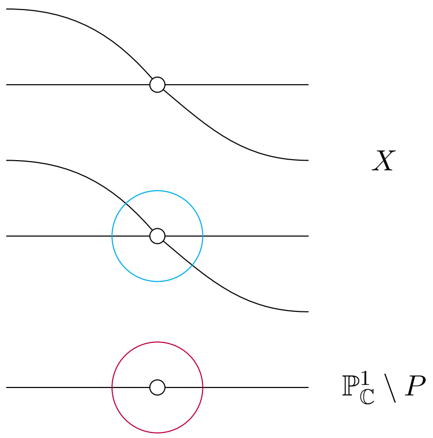
Definition 1.4.9. A **ramification type** is an equivalence class of triples $(G, P, \{C_p\}_{p\in P})$, where G is a finite group, P is a finite set of closed points of $\mathbb{P}^1(\mathbb{C})$ and C_p are conjugacy classes inside G. Two such triples $(G, P, \{C_p\}_{p\in P})$ and $(G', P', \{C'_p\}_{p\in P'})$ are equivalent if $P = P'$ and there exists an isomorphism of groups $G \to G'$ mapping each C_p to C'_p.

Theorem 1.4.10. *Riemann existence theorem- algebraic [16, theorem 1.2]*
Let $(G, P, \{C_p\}_{p\in P})$ be a ramification type with $P = \{p_1, ..., p_r\}$. Then, there exists a finite Galois extension $L/\mathbb{C}(x)$ with that ramification type if and only if there exists a set of generators of G, $\{g_1, ..., g_r\}$ with $g_i \in C_{p_i}$ for each i and $g_1 \cdot ... \cdot g_r = 1$.

Remark. Under some other assumptions on the ramification type it can be shown that such extension is unique. For more details see [16].

Theorem 1.4.11. *Riemann existence theorem- analytic [16, theorem 2.1]*
Let Y be a compact Riemann surface, $p_1, ..., p_s$ distinct points on it and $c_1, ..., c_s \in \mathbb{C}$. Then, there exists $g \in \mathcal{M}(Y)$ (the field of meromorphic functions on Y) with $g(p_i) = c_i$ for all i.

The connection between the two versions of the theorem is to be found in the theory of covers.



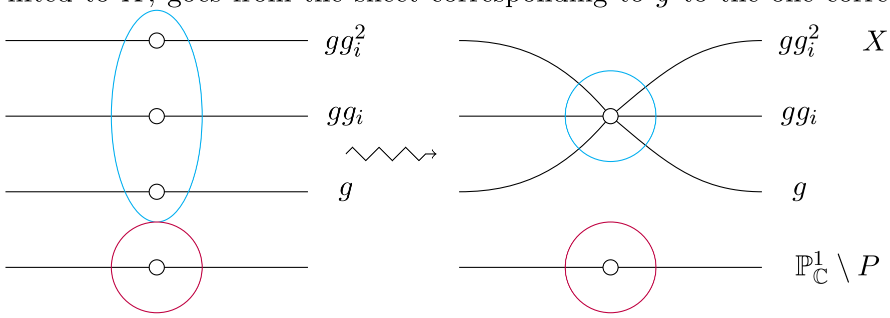
If $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$ is a Galois cover of Riemann surfaces, then we can extend it to the compactification of the surfaces and we get a ramified cover $\bar{\pi} : \bar{X} \to \mathbb{P}^1_{\mathbb{C}}$. Let's take a closer look at what happens. Consider a point $p_i \in P$, a "neighborhood" of it in $\mathbb{P}^1_{\mathbb{C}} \setminus P$ is a punctured disk D. Let E be a connected component of $\pi^{-1}(D)$, then $\pi_{|E} : E \to D$ is a homeomorphism (after possibly restricting D), so E is itself a punctured disk and $\pi_{|E}$ has to be of the form $z \mapsto z^e$ for an integer number e (all the homeomorphisms of the punctured disk can be put in this form). Thus, $\pi_{|E}$ has degree e and its group of deck transformations is generated by the map $\omega_e : z \mapsto \zeta_e z$ (where ζ_e is an e^{th} root of unity). Note that the choice of another connected component of $\pi^{-1}(D)$ corresponds to a conjugation of this element ω_e by a deck transformation of π. Hence, p_i defines a conjugacy class of G, call it C_{p_i}. When we consider the compactification of π it is clear that p_i becomes a ramified point with ramification index e. If $e = 1$, take the point p_i away from the set P.

Definition 1.4.12. In the setting above, we say that $(G = \text{deck}(\pi), P, \{C_p\}_{p \in P})$ is the **ramification type** of f.

Theorem 1.4.13. *Let $(G, P = \{p_1, ..., p_r\}, \{C_p\}_{p \in P})$ be a ramification type. Then, there exists a finite Galois covering space from a Riemann surface X, $\pi : X \to \mathbb{P}^1_{\mathbb{C}} \setminus P$, with that ramification type if and only if there exists a set of generators of G, $\{g_1, ..., g_r\}$ with $g_i \in C_{p_i}$ for each i and $g_1 \cdot ... \cdot g_r = 1$.*

Proof. If $\pi : X \to \mathbb{P}^1_{\mathbb{C}} \setminus P$ is a Galois cover, define $G = \text{deck}(\pi) = \frac{\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus P)}{\pi_1(X)}$. In particular G is a quotient of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus P)$, which is generated by loops γ_i around p_i such that $\prod_i \gamma_i = 1$. Denote by g_i the image of γ_i inside $\text{deck}(\pi)$, then, using the notations above, $g_{i|E}$ is a generator of $\text{deck}(\pi_{|E})$. In particular, it belongs to C_{p_i} and it has order e. As G is a quotient of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus P)$, these elements g_i generate G and they have the required properties.
Conversely, we have to construct a cover of Riemann surfaces with the prescribed ramification type. We can do it using a quotient of the universal cover of $\mathbb{P}^1_{\mathbb{C}} \setminus P$, but we will construct it in another way. Without loss of generality, we may assume that $p_r = \infty$. As a set, define $X := \mathbb{P}^1_{\mathbb{C}} \times G$ and as map of sets, define π to be the projection in the first component. Now, we need to give X a structure of Riemann surface. Let $g_i \in C_{p_i}$, define

"neighborhoods" of the missing point p_i such that the sheet corresponding to an element $g \in G$ is close to the sheet corresponding to gg_i and do it in a way such that the loop around p_i, when lifted to X, goes from the sheet corresponding to g to the one corre-



sponding to gg_i.

The fundamental group of the complex plane without $r-1$ points is the free group F_{r-1} on $r-1$ generators $\gamma_1, ..., \gamma_{r-1}$. Since G is generated by $g_1, ..., g_{r-1}$, it can be presented as a quotient of F_{r-1} by a normal subgroup, so it corresponds to a Galois covering space. Then $g_r = (g_1 \cdot ... \cdot g_{r-1})^{-1}$ corresponds to the loop going around ∞, $\gamma_r = (\gamma_1 \cdot ... \cdot \gamma_{r-1})^{-1}$. So, the Galois cover we constructed has the desired properties. qed

1.4.2 Proof

For proving the "classical" version of the conjecture 1.2.3, we can reduce to deal with only some particular cases. Here we prove the conjecture for each case using the tools developed in the previous discussion.

Remark. Let $\Delta = \sum_j \left(1 - \frac{1}{m_j}\right) P_j$ and $\Delta' = \sum_i \left(1 - \frac{1}{n_i}\right) P_i$ be two smooth orbifold divisors on a curve C, then we say that $\Delta | \Delta'$ if $\operatorname{supp}(\Delta)$ is contained in $\operatorname{supp}(\Delta')$ and, whenever $P_j = P_i$, then $m_j | n_i$. Note that, when we are in such situation we have that $(C, \Delta)^*(k, S) \subseteq (C, \Delta')^*(k, S)$.

Let g be the genus of the curve and call $s = \#\operatorname{supp}(\Delta)$.

- ⋆ If $g \geq 2$, then by Faltings' theorem $(C, 0)^*(k, S)$ is finite and it contains $(C, \Delta)^*(k, S)$, which is then finite as well.

Case deg$(K_C + \Delta) = 0$

- ⋆ If $\Delta = 0$, then $g = 1$ and we already have the result.
- ⋆ If $\Delta > 0$, then $g = 0$ so the degree of Δ must be exactly 2. Then $\deg(\Delta) = s - \sum_j (\frac{1}{m_j})$ and $s \geq 3$.

Case 1 $s = 3$, $\deg(\Delta) = s - \sum_j (\frac{1}{m_j})$, so $\sum_j \frac{1}{m_j} = 1$.

Case 2 $s = 4$, the only possibility is to have all $m_j = 2$.

- ⋆ $s > 4$, $\sum_j \frac{1}{m_j} \leq \frac{1}{2} s < s - 2$, so this case cannot happen.

Case deg($K_C + \Delta$)< 0

The genus can only be 0, so deg(Δ)< 2 and, as $\sum_j \frac{1}{m_j} \leq \frac{1}{2}s$, we deduce $s \leq 3$.

<u>Case 3</u> $s = 3$ with $\sum_{j=1,2,3} \frac{1}{m_j} > 1$

<u>Case 4</u> $s = 2$

⋆ $s = 1$, $\Delta = \left(1 - \frac{1}{m}\right)(a)$ for a point a of the curve, consider $\Delta' = \left(1 - \frac{1}{m}\right)(a) + \left(1 - \frac{1}{m}\right)(b)$ with b a point of the curve distinct from a, then $\Delta | \Delta'$, so by the remark above, we reduce this case to Case 4.

Case deg($K_C + \Delta$)> 0

<u>Case 5</u> If $g = 1$ and $s = 1$, $\Delta = \left(1 - \frac{1}{m}\right)(a)$, for a point a of the curve. Let p be a prime divisor of m (the case $m = \infty$ is already solved by Siegel's theorem) and define $\Delta' = \left(1 - \frac{1}{p}\right)(a)$, so that $\Delta' | \Delta$. So it is enough to prove the case with prime multiplicity.

⋆ If $g = 1$ and $s > 1$, consider Δ' constructed taking one of the points in Δ with exactly the same multiplicity. Then $\Delta' | \Delta$, so we have reduced this case to the previous one.

⋆ If $g = 0$, so deg(Δ)> 2, then $s > 2$. If $s > 3$, consider Δ' constructed using three of the points in supp(Δ) with their multiplicities so that $\Delta' | \Delta$. So we see it is enough to prove the statement for $s = 3$.

<u>Case 6</u> $g = 0$, $s = 3$, $\sum_{j=1,2,3} \left(1 - \frac{1}{m_j}\right) < 1$.

For each case 1,...,6, we construct classically orbifold étale covers so that, applying theorems 1.4.8 and 1.2.1 we will get the finiteness result as stated in the theorem 1.2.3. In most of the cases we work over the complex numbers to produce such covers and then we find covers over k in this way.

Step 1: The curve C is defined over a number field k, choose an embedding $i : k \hookrightarrow \mathbb{C}$, with an extension of scalars using this embedding ($C \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\mathbb{C})$) we can see C as a curve over $\mathbb{C}$ and so as a compact Riemann surface.

Step 2: Find a ramified cover with the desired properties of Riemann surfaces, $\pi : C' \to C$.

Step 3: As the Riemann surfaces involved are compact, the cover we found can be realized as an algebraic cover of algebraic varieties over the complex numbers.

Step 4: From the theory of algebraic covers, every algebraic cover over the complex numbers can be realized as a cover over $\bar{\mathbb{Q}}$, so we can see π as a ramified cover over $\bar{\mathbb{Q}}$.

Step 5: Choosing appropriate coordinates, we can express C, C' and π using a finite number of polynomials with coefficients in $\bar{\mathbb{Q}}$. Adding all of these coefficients to k, we obtain a finite extension k'/k and the morphism π can be defined over k'.

Case 1/ Case 6 For these cases we construct the cover using Riemann existence theorem 1.4.13. Indeed, consider $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, its fundamental group is generated by three loops $\gamma_0, \gamma_1, \gamma_\infty$ around the missing points. $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) = \langle \gamma_0, \gamma_1, \gamma_\infty | \gamma_0\gamma_1\gamma_\infty = 1\rangle$. Let p, q, r be the multiplicities "m_j" associated with $0, 1, \infty$ respectively. Let $\Gamma_{p,q,r}$ be the quotient of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ generated by three elements g_0, g_1, g_∞ such that $g_0^p = 1 = g_1^q = g_\infty^r = g_0 g_1 g_\infty$.

Lemma 1.4.14. *[9, lemma 2.1] In the notations above, $\Gamma_{p,q,r}$ is infinite and non abelian if and only if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$. Moreover, in this case, there exists a normal subgroup $H \leq \Gamma_{p,q,r}$, with $[\Gamma_{p,q,r} : H] < \infty$ and the orders of the classes corresponding to g_0, g_1, g_∞ in $\Gamma_{p,q,r}/H$ are respectively p, q, r.*

Using the above result from group theory, we can apply theorem 1.4.13 to the group $G := \Gamma_{p,q,r}/H$ to get a cover $\pi : C' \to C$ ramified only at $0, 1, \infty$ with orders p, q, r. Furthermore, using Riemann–Hurwitz' formula, we can compute the genus of the curve C':

$$2g(C') - 2 = d(0 - 2) + \epsilon_0(p - 1) + \epsilon_1(q - 1) + \epsilon_\infty(r - 1)$$

where $\epsilon_{0,1,\infty}$ are the number of points over $0, 1, \infty$ and d is the degree of the cover. Note that, since the cover is Galois, $\epsilon_0 = \frac{d}{p}, \epsilon_1 = \frac{d}{q}, \epsilon_\infty = \frac{d}{r}$, so that:

$$2g(C') = 2 - 2d + 3d - d\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right).$$

- Case 1: $g(C') = 1$ and $\pi(C'(k') \setminus \{\text{ramification points}\}) \subseteq (\mathbb{P}^1, \Delta)^*(S, k')$ for a finite extension k' of k by part 1 of theorem 1.4.8 and the former set is infinite by theorem 1.2.1, so the latter is infinite as well.

- Case 6: $g(C') \geq 2$ and $(\mathbb{P}^1, \Delta)^*(S, k) \subseteq \pi(C'(k'))$ for a finite extension k' of k by part 2 of theorem 1.4.8 and the latter is finite by Faltings' theorem 1.2.1 and then so is the former.

Example 1.4.15. Let's see some explicit examples of these covers for case 1.

1. Let $(p, q, r) = (3, 2, 6)$ and consider $C' = E$ the elliptic curve defined by the equation $y^2 z = x^3 - z^3$. Define the map:

$$\pi : E \to \mathbb{P}^1; \quad (x, y) \mapsto x^3.$$

This map has degree 6, is unramified outside $0, 1, \infty$ and the ramification can be computed as follows:

- over 0 we get the equations $x^3 = 0, y^2 = -1$, so we have two points with ramification order 3;

- over 1 we get the equations $x^3 = 1, y^2 = 0$, so we have three points with ramification order 2;

- around ∞ we can rewrite the map as $[x : y : z] \mapsto [x^3 : z^3] = [y^2 z + z^3 : z^3] = [y^2 + z^2 : z^2]$. Thus, over ∞ we get the equations $z^2 = 0, x^3 = 0$, so we have one point with ramification order 6.

2. Let $(p, q, r) = (4, 2, 4)$ and consider $C' = E$ the elliptic curve defined by the equation $y^2 z = x^3 - xz^2$. Define:

$$\pi : E \to \mathbb{P}^1; \quad (x, y) \mapsto x^2.$$

Then, $\deg(\pi) = 6$ and with similar computations as in the previous example we get that the map is ramified only over $0, 1, \infty$ with the prescribed ramification orders.

Case 2 Let $C' = E$ be an elliptic curve with equation $y^2 = x(x - 1)(x - \lambda)$, $\lambda \in k$, such that the points of two-torsion ($E[2]$) are contained in $E(k)$. Consider the projection map from the point at infinity:

$$\pi : E \to C; \quad (x, y) \mapsto x.$$

This map has degree 2 and is ramified only over $0, 1, \lambda, \infty$ with ramification order 2. Indeed, $\pi^{-1}(0, 1, \lambda, \infty)$ consists of the points of order 2 (and the unity of E, the point at infinity), the vertical line passing through these points is tangent to the curve and it meets E only at the given point and at infinity. So, this is the cover we were looking for. Thus, by theorem 1.4.8 part 1, $\pi(E(k) \setminus E[2]) \subseteq (\mathbb{P}^1, \Delta)^*(k, S)$ and the first set is infinite (possibly after a finite extension of the base field) by theorem 1.2.1, whence the result.

Case 3 In order to simplify the notations, call p, q, r the three multiplicities of the points in the orbifold divisor Δ. The condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, implies that $(p, q, r) \in \{(2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5)\}$.

- Case $(2, 2, r)$. Use the cover:

$$\pi : \mathbb{P}^1 \to \mathbb{P}^1; \quad t \mapsto \frac{t^r + t^{-r}}{2}$$

  which has degree $2r$ and is ramified over ± 1 with ramification order 2 and over ∞ with ramification order r.

- Cases $(2, 3, n)$. Consider the congruence subgroup $\Gamma(n) = \{A \in SL_2(\mathbb{Z}) | A \equiv_n \mathbb{1}_2\} \trianglelefteq SL_2(\mathbb{Z})$ and the modular curve associated with it: $X(n) = \mathbb{H}^*/\Gamma(n)$. We can realize these covers as the natural maps of the modular curves $X(n)$ to $X(1)$. From a discussion in [D-S], we can compute all the numbers we need. First of all the genus of these curves, for $n = 3, 4, 5$ (and 1) is always 0. $X(n)$ does not have elliptic points for our choices of n, so the ramification orders over the two elliptic points of $X(1)$, i and ρ are respectively 2 and 3. We have then more

ramification points only over the cusps and the ramification order at the cusps of $X(n)$ is $[SL_2(\mathbb{Z})_\infty : \pm\Gamma(n)_\infty] = n$ (index of the stabilizers of ∞).
For the cases $n = 3, 4$ explicit equations of these covers can be:

$$\pi_{(2,3,3)} : \mathbb{P}^1 \to \mathbb{P}^1; \quad t \mapsto \frac{(t^3+8)^3 t^3}{64(t^3-1)^3}$$

and

$$\pi_{(2,3,4)} : \mathbb{P}^1 \to \mathbb{P}^1; \quad t \mapsto \frac{-2^8 t(t^3-1)(t^3+8)^3}{(t^6-20t^3-8)^4}.$$

In all these cases we have: $\pi(\mathbb{P}^1 \setminus \{\text{ramification points}\}) \subseteq (\mathbb{P}^1, \Delta)^*(k, S)$, so, by theorem 1.4.8 part 1 and the fact that the first set is infinite, we get the conclusion.

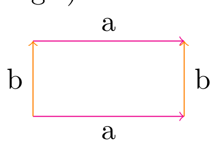
Case 4 Without loss of generality we can assume that the support of Δ consists of the two points $0, \infty$. If $\Delta = \left(1 - \frac{1}{m}\right)(0) + \left(1 - \frac{1}{m}\right)(\infty)$ we can simply use the cover:

$$\pi : \mathbb{P}^1 \to \mathbb{P}_{;} \quad t \mapsto t^m$$

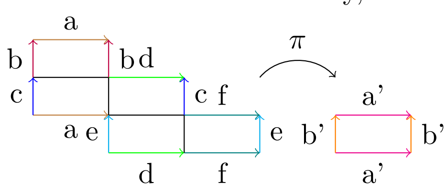
which is ramified only over 0 and ∞ with ramification order m. Otherwise, if $\Delta = \left(1 - \frac{1}{m}\right)(0) + \left(1 - \frac{1}{n}\right)(\infty)$, define $\Delta' = \left(1 - \frac{1}{mn}\right)(0) + \left(1 - \frac{1}{mn}\right)(\infty)$, then $\Delta | \Delta'$ and the multiplicities of the two points in Δ' are the same, as in the previous case. So, $\pi(\mathbb{P}^1 \setminus \{0, \infty\}) \subseteq (\mathbb{P}^1, \Delta')^*(k, S) \subseteq (\mathbb{P}^1, \Delta)^*(k, S)$ and, from the fact that the first set is infinite, we get that also the last one is infinite.

Case 5 When p is an odd prime, we can use the theory of "origamis". Let's start with an example to understand the idea.

Example 1.4.16. Consider $p = 5$. Since $g = 1$, C is an elliptic curve and, on the complex numbers, it can be represented as a quotient $\mathbb{C}/\Lambda$, for $\Lambda = \tau\mathbb{Z} + \mathbb{Z}$, with τ a complex number with positive imaginary part. Graphically, we can represent it as a parallelogram with the opposite edges identified (for simplicity represented with a rectangle).



Construct C' glueing together p copies of this rectangle as in the following picture and define π in the obvious way, as the identity in each rectangle:

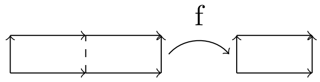


From the picture it is clear that π has order 5 and is ramified only at the point corresponding to all the vertices (which are identified) with order 5. To compute the genus of the curve C', we can use Euler's formula: $2 - 2g(C') = F - E + V$ (where F is the number of faces, E the number of edges and V the number of vertices). In our case we have then $g(C') = \frac{1}{2}(-5 + 10 - 1 + 2) = 3$.

The general situation is exactly the same, considering a similar picture with p rectangles in the shape of a stair. The cover $\pi : C' \to C$ so constructed has degree p and is ramified only in one point with order p. Moreover, the genus of the curve C' can be computed as before: $F = p,\ E = 2p,\ V = 1$, so $g(C') = \frac{1}{2}(-p+2p-1+2) = \frac{1}{2}(p+1) \geq 2$.

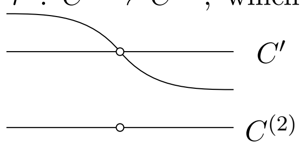
Remark. Actually, there are explicit equations for the curve C' and the morphism π for every degree. See [14, theorem 3].

Whereas, if $p = 2$, we cannot do this construction. We will construct the ramified cover in two steps.

Step 1 Construct $f : C^{(2)} \to C$ a double unramified cover of C. In the picture a representation of a possible cover. 

Step 2 Construct a cover of $C^{(2)}$ ramified with order 2 over two points a_1, a_2. We do it with a construction similar to the one we used for Riemann existence theorem. Consider the fundamental group $\pi_1(C^{(2)} \setminus \{a_1, a_2\}) = \langle \alpha, \beta, \gamma_1, \gamma_2 | \alpha\beta\alpha^{-1}\beta^{-1} = \gamma_1\gamma_2 \rangle$, where γ_i are loops around a_i and α, β are the usual loops that generate the fundamental group of an elliptic curve. Consider the homomorphism of groups:

$$\varphi : \pi_1(C^{(2)} \setminus \{a_1, a_2\}) \to S_2 = \{1, \sigma\} \quad \alpha, \beta \mapsto 1,\ \gamma_i \mapsto \sigma.$$

This induces an unramified cover of Riemann surfaces: $r : \tilde{C}' \to C^{(2)}$, which, following the construction in theorem 1.4.13, looks like: 

Then, compactifying domain and codomain, we obtain a morphism $\rho : C' \to C^{(2)}$ with ramification of order 2 over the two points a_1, a_2. Moreover, with Riemann–Hurwitz' formula, we can compute the genus of the curve C':

$$2g(C') - 2 = 2(2 - 2) + 2(2 - 1) \Rightarrow g(C') = 2.$$

Finally, compose $\rho \circ f = \pi$ choosing a_1 and a_2 over the same point α in C. So, π is a map of compact Riemann surfaces of degree 4 and ramified only over α with order 2. This is the map we were looking for.

In all the cases discussed above, we have found a classically étale orbifold cover of the elliptic curve C from a curve C' with genus ≥ 2. So, applying part 2 of the theorem 1.4.8, we get that the orbifold integral points are finitely many:

$$(C, \Delta)^*(S, k) \subseteq \pi(C'(k')) \quad \text{which is finite by Faltings' theorem 1.2.1.}$$

This concludes the proof of the classical version of the conjecture for curves. qed

1.5 Rational and integral points on surfaces

In the last sections of this chapter, we discuss some known results about surfaces. The Italian school at the end of the 19$^{\text{th}}$ century classified surfaces birationally, thus, we can study rational and integral points on surfaces looking at each case. Density of rational points is well-understood in most cases; on the other hand, the behavior of integral points is still an open problem. In the next sections we see an analogue of Siegel's theorem 1.2.2, which, however, does not characterize completely mordellicity.

We start by presenting the classification, every (complex) algebraic surface is birational to a surface in one (or more, they are not exclusive) of the following classes.

- **Rational surfaces**: these are birationally isomorphic to the plane. Clearly, they are potentially dense.

- **Ruled surfaces**: they are birationally isomorphic to a product $C \times \mathbb{P}^1$, where C is a curve. They are potentially dense if and only if C is so, otherwise they are neither potentially dense, nor mordellic.

- **Elliptic surfaces**: they admit a dominant map f towards a curve, whose general fiber has genus one. They can be potentially dense, for example if f admits a section with infinite order (with respect to the group law of the fibers).

- **Abelian surfaces**: surfaces that are also abelian varieties. They admit closed algebraic points which generate a dense subgroup, therefore they are potentially dense.

- **K3 surfaces**: they are (smooth projective) surfaces which are simply connected and whose canonical bundle is trivial. It is conjectured that they are potentially dense, but it is proven only in some cases, with additional information.

- **Kummer, bielliptic (or hyperelliptic) surfaces**: they are quotients of abelian surfaces. They are potentially dense since they are dominated by abelian surfaces.

- **Surfaces of general type**: their canonical divisor is big. According to Bombieri–Lang–Vojta's conjecture they are mordellic.

Remark. What is proven for surfaces agrees with some conjectures we will present in the next chapters for higher dimensional varieties, which use the Kodaira dimension (defined later).

1.5.1 Numerical properties

The goal now is to generalize Siegel's theorem to the case of surfaces, with the same approach used for curves, applying the subspace theorem 1.3.1. However, we need to be careful, the divisors now are sums of curves and the dimensions of the analogue of the spaces in the filtration $W_{\nu,j}$ depend on the curve and also on j, there is no uniform upper bound. To be able to apply the subspace theorem, we need to add some more conditions on the type of divisor we are removing from the surface. These are numerical conditions, so we start by presenting the tools we need to understand them. In particular, here we discuss general intersection numbers, numerical equivalence and some numerical properties on surfaces.

In algebraic geometry a tool to control intersections is the Hilbert polynomial. The idea to define general intersection numbers is to extend the notion of Hilbert polynomial for a sum of divisors.

Definition 1.5.1. Recall that, for a proper scheme X over a field k and a coherent sheaf $\mathscr{F}$ on it, we can define the **Euler characteristic** of $\mathscr{F}$ as the integer:

$$\chi(X, \mathscr{F}) := \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathscr{F}).$$

Theorem 1.5.2. *[D, theorem 1.5] Let $D_1, ..., D_r$ be Cartier divisors on a proper scheme X over a field k and let $\mathscr{F}$ be a coherent sheaf on X. The function*

$$(m_1, ..., m_r) \mapsto \chi(X, \mathscr{F}(m_1 D_1 + ... + m_r D_r))$$

takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients of degree at most the dimension of the support of $\mathscr{F}$.

Definition 1.5.3. Let $D_1, ..., D_r$ be Cartier divisors on a proper scheme X over a field k with $r \geq \dim(X)$. The coefficient of $m_1 \cdot ... \cdot m_r$ in the polynomial $\chi(X, m_1 D_1 + ... + m_r D_r)$ is called the **intersection number** of the divisors and is denoted by $D_1 \cdot ... \cdot D_r$.
If Y is a subscheme of X of dimension at most s, we set

$$D_1 \cdot ... \cdot D_s \cdot Y := D_{1|Y} \cdot ... \cdot D_{s|Y}.$$

Proposition 1.5.4. *[D, proposition 1.8] Let $D_1, ..., D_n$ be Cartier divisors on a proper scheme X of dimension n such that D_n is effective with associated subscheme Y. Then:*

$$D_1 \cdot ... \cdot D_n = D_1 \cdot ... \cdot D_{n-1} \cdot Y.$$

Proposition 1.5.5. *[H, ch. V, exercise 1.1] Let D, D' be divisors on a surface X, then*

$$D \cdot D' = \chi(X, -D - D') - \chi(X, -D) - \chi(X, -D') + \chi(X, \mathcal{O}_X).$$

Proposition 1.5.6. *If D is a divisor on a surface X and $C \subseteq X$ is a curve, things are simple:*

$$D \cdot C = \deg(D_{|C}).$$

Remark. If the divisor and the curve are in sufficiently general positions, from this result we can see that their intersection number is exactly the number of intersection points with the muliplicities coming from the coefficients in D.

Proof. We can assume that the curve is normal, otherwise we can change the surface birationally such that the curve becomes so. We assume this implicitly whenever it is needed, from now to the end of the chapter. Consider the exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and the twisted sequence

$$0 \to \mathcal{O}_X(-C-D) \to \mathcal{O}_X(-D) \to \mathcal{O}_C(-D_{|C}) \to 0.$$

From them we get the equalities $\chi(X,-C) - \chi(X,\mathcal{O}_X) + \chi(C,\mathcal{O}_C) = 0$ and $\chi(X,-C-D) - \chi(X,D) + \chi(C,D_{|C}) = 0$. Then, putting together these two equations and the formula from proposition 1.5.5 with $D' = C$, we get:

$$D \cdot C = -\chi(C,-D_{|C}) + \chi(C,\mathcal{O}_C) = \deg(D_{|C})$$

where the last equality comes from Riemann–Roch theorem [H, ch.IV, theorem 1.3]. qed

Also ampleness (see the first section 2.1.1 in the next chapter for the definition) can be characterized in numerical terms, at least on surfaces.

Theorem 1.5.7. *Nakai–Moishezon criterion [H, ch.V, theorem 1.10]*
A divisor D on a surface X is ample if and only if $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C in X.

Lemma 1.5.8. *Let D be an ample divisor on a nonsingular projective surface X over a field k, then for positive integers N, we have:*

$$h^0(ND) = \frac{N^2D^2}{2} + O(N).$$

Proof. Let K be the canonical divisor of X, then Riemann–Roch theorem for surfaces ([H, ch.V, theorem 1.6]) gives:

$$h^0(X,ND) = \frac{1}{2}(ND)^2 - \frac{1}{2}(ND \cdot K) + \chi(X,\mathcal{O}_X) + h^1(X,ND) - h^0(K-ND).$$

For N large enough, $K - ND < 0$, so $h^0(K-ND)$ vanishes. Moreover, as D is ample and $\mathcal{O}_X$ is coherent, for $N \gg 0$, $h^1(X,ND) = h^1(X,\mathcal{O}_X \otimes \mathcal{O}_X(ND)) = 0$ ([H, ch.III, proposition 5.3]). The term $\chi(X,\mathcal{O}_X)$ is a constant and $ND \cdot K$ is linear in N. Thus, we have the claim. qed

Lemma 1.5.9. *Let D be a divisor and C a curve on a smooth projective surface X over a field k, then:*

$$h^0(X,D) - h^0(X,D-C) \leq \max\{0, 1 + D \cdot C\}.$$

Proof. The exact sequence

$$0 \to \mathcal{O}_X(-C+D) \to \mathcal{O}_X(D) \to \mathcal{O}_C(D_{|C}) \to 0$$

gives a long exact sequence in cohomology, from which we get an injection of vector space

$$0 \to H^0(X,D)/H^0(D-C) \to H^0(C,D_{|C}).$$

From this we get:

$$h^0(X,D) - h^0(X,D-C) \leq \dim H^0(C,D_{|C}).$$

Now, if D is not effective, then $\dim H^0(C,D_{|C}) = 0$. Otherwise, if D is effective, note that $H^0(C,K_C-D) \subseteq H^0(C,K_C)$, where the inclusion is given by multiplication by an element of $H^0(C,D) \neq 0$. Thus, applying Riemann–Roch theorem, we get:

$$\begin{aligned} h^0(C,D_{|C}) &= D\cdot C + 1 - g + h^0(C,K_C - D) \\ &\leq D\cdot C + 1 - g + h^0(C,K_C) = D\cdot C + 1 \end{aligned}$$

where g denotes the genus of C. qed

Lemma 1.5.10. *Let D be an ample effective divisor on a smooth projective surface X over a field k, C an irreducible component of D. For positive integers N, j we have that either $H^0(X, ND-jC) = \{0\}$, or*

$$0 \leq h^0(ND-jC) - h^0(ND-(j+1)C) \leq N(D\cdot C) - jC^2 + 1.$$

Proof. Suppose that $(ND-jC)\cdot C \geq 0$, in this case we can apply lemma 1.5.9 with the divisor $ND-jC$ to get the result.
Otherwise, if $(ND-jC)\cdot C < 0$, we claim that $\mathcal{O}_X(ND-jC)$ has no regular sections. Indeed, if it had, there would be an effective divisor E linearly equivalent to $ND-jC$ (see proposition 2.1.5), thus $E\cdot C = (ND-jC)\cdot C < 0$. However, we can write $E = E' + rC$, where $E' \geq 0$ does not contain C in its support and $r \geq 0$, then $E\cdot C = E'\cdot C + rC^2$. As E' does not contain C, $E'\cdot C \geq 0$ is the number of intersection points of the curve with the divisor. Besides, $D^2 > 0$ as well as $D\cdot C > 0$ because D is ample (by Nakai–Moishezon criterion 1.5.7), so, to have $(ND-jC)\cdot C < 0$, C^2 must be positive. Therefore, $E\cdot C = E'\cdot C + rC^2 > 0$, contradiction. qed

Lemma 1.5.11. *Let D be an ample divisor, C an effective curve on a smooth projective surface X over a field k. Then*

$$D^2C^2 \leq (D\cdot C)^2.$$

Proof. As D is ample, by Nakai–Moishezon criterion 1.5.7, $D^2 > 0, D\cdot C > 0$. Hence the inequality is non-trivial only when $C^2 > 0$. In this case, suppose the conclusion of the theorem did not hold. Consider the intersection form on the subgroup generated by D and C inside $\mathrm{Pic}(X)$, we claim that it would be positive definite. Indeed, the discriminant of the inequality

$$(xD + yC)^2 = x^2D^2 + 2xyD \cdot C + y^2C^2 > 0$$

would be negative, which gives the claim. However, by Hodge index theorem [H, ch.V, theorem 1.9 and remark 1.9.1] this form cannot be positive definite, whence a contradiction and the conclusion of the proof. qed

1.5.2 An analogue of Siegel's theorem for surfaces

Now, we have the technical tools we need to state and prove the result that generalizes Siegel's theorem (approached using the subspace theorem 1.3.1) to surfaces.

During the proof we use two lemmas, which we state here.

Lemma 1.5.12. *Let $x_1, ..., x_h, U_1, ..., U_h \geq 0$ and $R \leq h$ be integers such that $\sum_{j=1}^{R} U_j \leq d$. Suppose further that for each j, $0 \leq x_j \leq U_j$ and $\sum_{j=1}^{h} x_j = d$. Then:*

$$\sum_{j=1}^{h} jx_j \geq \sum_{j=1}^{R} jU_j.$$

Proof. We can do the following computations:

$$\begin{aligned}\sum_{j=1}^{R} jU_j + (R+1)d - \sum_{j=1}^{h} jx_j &= \sum_{j=1}^{R} jU_j + \sum_{j=1}^{h} (R+1-j)x_j \\ \leq \sum_{j=1}^{R} jU_j + \sum_{j=1}^{R} (R+1-j)x_j &\leq \sum_{j=1}^{R} jU_j + \sum_{j=1}^{R} (R+1-j)U_j = (R+1)\sum_{j=1}^{R} U_j.\end{aligned}$$

Whence $\sum_{j=1}^{h} jx_j \geq \sum_{j=1}^{R} jU_j + (R+1)(d - \sum_{j=1}^{R} U_j)$ and the result follows since the last parenthesis is ≥ 0 by assumption. qed

Lemma 1.5.13. *Let V be a vector space of finite dimension d over a field k. Let $V = W_1 \supset W_2 \supset ... \supset W_h$ and $V = W_1^* \supset W_2^* \supset ... \supset W_{h^*}^*$ be two filtrations on V. Then, there exists a basis $\psi_1, ..., \psi_d$ of V which contains a basis of each W_j and each W_j^*.*

Proof. We proceed by induction on d. If $d = 1$ the statement is obvious. Then, we can suppose (by possibly refining the first filtration), that W_2 is an hyperplane in V. Let $W_i' := W_i^* \cap W_2$. By the inductive hypothesis there exists a basis $\psi_1, ..., \psi_{d-1}$ of W_2 containing basis of both $W_2, ..., W_h$ and $W_1', ..., W_{h^*}'$. If all the $W_i^* = W_i'$ for $i \geq 2$, then we can just complete the set $\{\psi_1, ..., \psi_{d-1}\}$ to any basis of V.
Otherwise, let l be the minimum index such that $W_l^* \not\subseteq W_2$ and let $\psi_d \in W_l^* \setminus W_2$. For sure $\{\psi_1, ..., \psi_d\}$ contains a basis for each W_i. Now, let $i \in \{1, ..., h^*\}$, if $i > l$, $W_i^* = W_i'$, so the basis we have constructed contains a basis of W_i. On the other hand, if $i \leq l$, the constructed basis contains $\psi_d \in W_l^* \subset W_i^*$ and it contains a basis for W_i', which is an hyperplane in W_i^*, thus it contains a basis of W_i^*. qed

Theorem 1.5.14. *Let $\widetilde{X}$ be a smooth projective surface over a number field k and let $X \subseteq \widetilde{X}$ be an affine open subset. Assume that $\widetilde{X} \setminus X = D_1 \cup ... \cup D_r$, $r \geq 2$, where the D_i's are distinct irreducible divisors such that:*

(i) no three of them share a common point;

(ii) there exists positive integers $p_1, ..., p_r$ such that $D := p_1 D_1 + ... + p_r D_r$ is ample and the following holds. Let ξ_i, for $i = 1, ..., r$ be the minimal positive solution of the equation $D_i^2 \xi^2 - 2(D \cdot D_i)\xi + D^2 = 0$ (this solution always exists), we have the inequality

$$2D^2 \xi_i > (D \cdot D_i)\xi_i^2 + 3D^2 p_i.$$

Then there exists a curve on X containing all the $(S, \Delta := D_1 + ... + D_r)$-integral points, where S is a finite set of non-archimedean places of k.

Proof. By enlarging k, if necessary, we may assume all the D_i's are defined over k. Let S' be the union of S and the archimedean places of k.

Step 1 We reduce to prove the following.
For every infinite sequence of (S, Δ)-integral points on X, there exists a curve defined over k, containing an infinite subsequence.
Indeed, assume this holds and enumerate all curves on X defined over k: $C_1, C_2, ...$ Remark that X has at most countably many curves on it. In fact, given $N, d \in \mathbb{N}$, there are countably many polynomials with degree smaller than d and coefficients in k smaller than N. Thus, there are at most countably many polynomials that define curves on X.
If the conclusion of the theorem was not true, then for every curve, there existed an (S, Δ)-integral point outside that curve. Thus, we can construct a sequence of (S, Δ)-integral points by taking $P_1 \notin C_1$, $P_2 \notin C_1 \cup C_2$, and so on ($P_n \notin C_1 \cup ... \cup C_n$). From this sequence we cannot extract any infinite subsequence contained in a single curve. Contradiction.

Step 2 To prove the above claim, we want to follow the steps done for Siegel's theorem, reducing to the subspace theorem 1.3.1, but this time we have to be more careful. Let $\{P_i\}_{i \in \mathbb{N}}$ be an infinite sequence of distinct (S, Δ)-integral points, by possibly passing to a subsequence, we may assume that it converges for all $\nu \in S'$. Call the ν-adic limit $R_\nu \in \widetilde{X}(k_\nu)$. Three different situations can happen (and no others because of assumption (i)).

Case A The point $R_\nu \notin \mathrm{Supp}(D)$.

Case B There exists an index i such that $R_\nu \in \mathrm{Supp}(D_i)$, but for all $j \neq i$, $R_\nu \notin \mathrm{Supp}(D_j)$. In this case denote by $D_\nu := D_i$ and $p_\nu := p_i$.

Case C The point R_ν belongs exactly to two components D_i and D_j. Denote $D_\nu := D_i$, $D_\nu^* := D_j$ and $p_\nu := p_i$, $p_\nu^* := p_j$.

The goal of this step is to prove the inequality that allows us to apply the subspace theorem 1.3.1. More precisely, fix $N \gg 0$ and let $V_N := H^0(\widetilde{X}, ND)$ (where $\widetilde{X}$ is considered over k). Note that each function is regular in the affine X, in particular, it can be expressed as a polynomial in the affine coordinates. Let $f_1, ..., f_M$ be a basis of V_N (for $N \gg 0$, $M \geq 2$). By multiplying all f_j's by a suitable constant, we may assume that all $f_j(P_i) \in \mathcal{O}_{k,S}$. In each of the cases above, for every $\nu \in S'$ we want to construct linear functions in the f_j's, $L_{\nu,j}$ which are linearly independent and such that for an infinite subsequence of $\{P_i\}_{i\in\mathbb{N}}$, there exists $\mu_\nu > 0$ and a constant α_ν independent of i such that:

$$\prod_{j=1}^{M} |L_{\nu,j}(P_i)|_\nu \leq \alpha_\nu (\max_j |f_j(P_i)|_\nu)^{-\mu_\nu}.$$

<u>Case A</u> Set $L_{\nu,j} := f_j$. Since the limit R_ν is finite, these functions are bounded on the sequence. Therefore:

$$\prod_{j=1}^{M} |L_{\nu,j}(P_i)|_\nu = \prod_{j=1}^{M} |f_j(P_i)|_\nu \frac{\max_j |f_j(P_i)|_\nu}{\max_j |f_j(P_i)|_\nu} \leq \alpha_\nu (\max_j |f_j(P_i)|_\nu)^{-1}.$$

<u>Case B</u> Choose a local equation t_ν at R_ν for the divisor D_ν. Define a filtration of $V_N = W_1 \supseteq W_2 \supseteq ...$ with

$$W_j := \{f \in V_N | \mathrm{ord}_{D_\nu} f \geq j - 1 - Np_\nu\},\ j \geq 1.$$

Choose a basis of V_N containing a basis of the filtration, call these elements $L_{\nu,j}$ and note that they can be written as linear combinations of the f_j's. In particular, in this new basis, there are $\dim(W_j/W_{j+1})$ elements with $\mathrm{ord}_{D_\nu} = j - 1 - Np_\nu$, whence:

$$\sum_{j=1}^{M} \mathrm{ord}_{D_\nu} L_{\nu,j} = \sum_{j=1}^{M} (j - 1 - Np_\nu) \dim(W_j/W_{j+1}).$$

We want to apply lemma 1.5.12, so we need to set x_j, U_j, h, R and we need to check the conditions in the statement. Let $x_j := \dim(W_j/W_{j+1})$ and let h be the maximum index such that $W_j \neq 0$, note that $\sum_{j=1}^{h} x_j = \dim V_N = M = \frac{N^2D^2}{2} + O(N)$, where the last equality comes from lemma 1.5.8. Define $U_j := 1 + N(D \cdot D_\nu) - jD_\nu^2$. By lemma 1.5.10, $0 \leq x_j \leq U_j$.
Let ξ be the minimal positive solution to the equation:

$$D_\nu^2 \xi^2 - 2(D \cdot D_\nu)\xi + D^2 = 0 \tag{$\star$}$$

(so $\xi = \xi_i$ in the statement). By lemma 1.5.11, the solutions of this equation are real. Call ζ the other solution, then $\zeta + \xi = \frac{2D \cdot D_\nu}{D_\nu^2}$ and $\zeta\xi = \frac{D^2}{D_\nu^2}$. As D is ample by assumption, using Nakai–Moishezon criterion 1.5.7, we see that

$D^2 > 0$ and $D \cdot D_\nu > 0$, therefore at least one of the two roots must be positive. If $D_\nu^2 < 0$, then, it is clear that $\xi D_\nu^2 \leq D \cdot D_\nu$. If, instead, $D_\nu^2 \geq 0$, then ξ and ζ are positive as both the sum and the product of the two roots are positive and, as ξ is the smallest root, from $\zeta + \xi = \frac{2D \cdot D_\nu}{D_\nu^2}$, we deduce that $\xi D_\nu^2 \leq D \cdot D_\nu$. Choose $0 < \lambda < \xi$ such that:

$$\lambda^2 \frac{D \cdot D_\nu}{2} - \lambda^3 \frac{D_\nu^2}{3} - \frac{D^2 p_\nu}{2} > 0.$$

This is possible by continuity, because ξ satisfies the same inequality. Indeed, by condition (ii), $2D^2\xi > D \cdot D_\nu \xi^2 + 3D^2 p_\nu$, putting together this with $(\star)$ multipled by 2ξ, gives the result. Moreover, since $\lambda < \xi$ and ξ is the minimal positive solution that satisfies $(\star)$,

$$\frac{D_\nu^2 \lambda}{2} - D \cdot D_\nu \lambda + \frac{D^2}{2} > 0.$$

Now, set $R := \lfloor \lambda N \rfloor$. Let us verify the conditions to apply lemma 1.5.12:

$$\begin{aligned}
\sum_{j=1}^{R} U_j &= RN(D \cdot D_\nu) + \mathbb{R}^2 \frac{D_\nu^2}{2} + O(R+N) \\
&\leq N^2 (D \cdot D_\nu \lambda - \frac{D_\nu^2 \lambda^2}{2}) + O(N) \\
&< N^2 \frac{D^2}{2} + O(N) = M = \sum_{j=1}^{h} x_j.
\end{aligned}$$

Note that, for $j \leq R$, $U_j > 0$, as $0 \leq D \cdot D_\nu - \xi D_\nu^2 < D \cdot D_\nu - \lambda D_\nu^2$. This, in turn, implies $R \leq h$, otherwise, if not, $\sum_{j=1}^{R} U_j > M$, contradiction. Thus, lemma 1.5.12 yields:

$$\begin{aligned}
\sum_{j=1}^{h} j x_j \geq \sum_{j=1}^{R} U_j& \\
&= \sum_{j=1}^{R} j(1 + N(D \cdot D_\nu) - j D_\nu^2) \\
&= \frac{R(R+1)}{2}(1 + N(D \cdot D_\nu)) - \frac{R(R+1)(2R+1)}{6} D_\nu^2 \\
&= N^3 \left(\frac{\lambda^2 D \cdot D_\nu}{2} - \frac{\lambda^3 D_\nu^2}{3} + O(\frac{1}{N}) \right).
\end{aligned}$$

Hence:

$$\begin{aligned}\sum_{j=1}^{M} \mathrm{ord}_{D_\nu} L_{\nu,j} &= \sum_{j=1}^{h} (j-1-Np_\nu)x_j \\ &\geq \sum_{j=1}^{h} jx_j - (Np_\nu+1)M \\ &\geq N^3\left(\frac{\lambda^2 D\cdot D_\nu}{2} - \frac{\lambda^3 D_\nu^2}{3} - \frac{D^2 p_v}{2} + O(\frac{1}{N})\right) > 0 \quad \text{for } N \gg 0\end{aligned}$$

We can write each $L_{\nu,j} = t_\nu^{\mathrm{ord}_{D_\nu} L_{\nu,j}} F_j$, where F_j has non-negative order at D_ν, so $|F_j(P_i)|_\nu$ are bounded for $i \gg 0$. Hence, there exists a positive constant β such that

$$\prod_{j=1}^{M} |L_{\nu,j}(P_i)|_\nu \leq \beta |t_\nu(P_i)|_\nu^{\sum_{j=1}^{M} \mathrm{ord}_{D_\nu} L_{\nu,j}}.$$

In the same way, there exists a constant γ such that, for $i \gg 0$,

$$\max_j |f_j(P_i)|_\nu \leq \gamma |t_\nu(P_i)|_\nu^{-Np_\nu}.$$

Putting together the last two estimates, we finally get the claim, using that $\sum_{j=1}^{M} \mathrm{ord}_{D_\nu} L_{\nu,j} > 0$.

Case C Consider two filtrations on V_N:

$$\begin{aligned} W_j &:= \{f \in V_N | \mathrm{ord}_{D_\nu} f \geq j-1-Np_\nu\} \\ W_j^* &= \{f \in V_N | \mathrm{ord}_{D_\nu^*} f \geq j-1-Np_\nu^*\}.\end{aligned}$$

Using lemma 1.5.13, we can construct a basis for V_N that contains a basis for each W_j, W_j^*. The elements of this basis can be written as linear combinations $L_{\nu,j}$ of the f_j's. As $\widetilde{X}$ is smooth, the local ring at R_ν is an UFD of dimension 2, thus, if we choose t_ν and t_ν^* local equations for D_ν and D_ν^*, they generate the local ring and they are coprime. Therefore, $L_{\nu,j} = t_\nu^{\mathrm{ord_{D_\nu} L_{\nu,j}}} t_\nu^{*\mathrm{ord}_{D_\nu^*} L_{\nu,j}} F_j$, where F_j is regular at R_ν, so it is bounded on all but finitely many of the P_i's. Using the same calculations as in case B, we get the claim.

As the constant function $1 \in V_N$, $\max_j |f_j(P_i)|_\nu > 0$. Define $\mu := \min_{\mu \in S'} \mu_\nu$, so that

$$\prod_{j=1}^{M} |L_{\nu,j}(P_i)|_\nu \leq \alpha_\nu (\max_j |f_j(P_i)|_\nu)^{-\mu} \quad \forall \nu \in S'.$$

Step 3 Now, we want to apply the subspace theorem 1.3.1. Let $\mathbf{x}_i := [f_1(P_i) : \dots : f_M(P_i)]$, we may assume that the height $H(\mathbf{x}_i)$ goes to infinity. Indeed, if not, it would be bounded for $i \gg 0$ and the points $\mathbf{x}_i$ would lie in a finite set. This would imply

that the function f_1/f_2 is constant $= c$ on an infinite subsequence, which means that an infinite subsequence lies in the curve defined by the equation $f_1 - cf_2 = 0$, so the claim is true in this case.
But, if $H(\mathbf{x}_i)$ goes to infinity, for all, but finitely many i's, $H(\mathbf{x}_i)^{-\frac{\mu}{2}} \prod_{\nu\in S'} \alpha_\nu \leq 1$.
Therefore

$$\prod_{\nu\in S'} \prod_{j=1}^{M} |L_{\nu,j}(P_i)|_\nu \leq H(\mathbf{x}_i)^{-\frac{\mu}{2}},$$

which is what we need to apply the subspace theorem 1.3.1. We finally get a non-trivial relation $\sum_j a_j f_j(P_i) = 0$ on an infinite subsequence of $\{P_i\}_{i\in\mathbb{N}}$. As the functions f_j are linearly independent, this relation defines the desired curve.

qed

Corollary 1.5.15. *Let $\widetilde{X}$ be a smooth projective surface and $X \subseteq \widetilde{X}$ an affine open subset such that $\widetilde{X} \setminus X = D_1 \cup ... \cup D_r$, where the D_i are distinct irreducible divisors such that no three of them share a common point. Assume, moreover, that $r \geq 4$ and that there exists positive integers $p_1, ..., p_r, c$ such that $p_i p_j (D_i \cdot D_j) = c$ for all pairs i, j. Then, there exists a curve on X containing all the $(S, \Delta := D_1 + ... + D_r)$-integral points.*

Remark. This corollary tells us that, if we remove more than 4 curves from a surface, this is "most likely" mordellic, this can be seen as an analogue of what Siegel's theorem says. It is believed that taking out a divisor of sufficiently large degree from a projective variety produces a variety of general type, which is conjectured to be mordellic (we discuss in the next chapters this conjecture).

Proof. We have to check that, with the p_i's in the statement, the assumptions of theorem 1.5.14 are satisfied. Note that

$$D \cdot D_i = \sum_{j=1}^{r} p_j D_j \cdot D_i = \frac{cr}{p_i}; \qquad D^2 = \sum_{i,j} p_i p_j D_i \cdot D_j = r^2 c; \qquad D_i^2 = \frac{c}{p_i^2}.$$

Thus, $\xi_i = rp_i$ and the inequality becomes $2r^3cp_i > r^3cp_i + 3r^2cp_i$, which is equivalent to $r \geq 4$. qed

Chapter 2

Kodaira dimension

In higher dimension there are varieties which are neither potentially dense nor mordellic. The objective would be to identify the conditions under which an orbifold pair has one of these two properties and, for general varieties, to construct the **core map**. This fibration conjecturally separates the mordellic part (base of the fibration) and the potentially dense part (the fibers of the fibration) of a variety.

The invariant which will play the role of the genus for higher dimensional varieties is the Kodaira dimension. Its definition and some of its properties are discussed in the second section.

The first part of the chapter, instead, gives some preliminary results about the two main objects that appear in the definition of the Kodaira dimension: linear systems and canonical sheaves.

The third section deals with a property of the Kodaira dimension which relates the dimension of two varieties in a fibration and the dimension of the fibers: the easy additivity property. This will be used later to study the core map.

In the last two sections we present two fibrations, the Iitaka fibration and the MRC quotient. The main property of the first one is that its base has dimension equal to the Iitaka dimension of the variety and its fibers have zero Kodaira dimension. The second one, instead, is characterized by the fact that its base has positive Kodaira dimension and its fibers are rationally connected. The core map will turn out to be a composition of appropriate orbifold modifications of these two maps.

2.1 Preliminaries

2.1.1 Divisors and embeddings

In this first part of the section we discuss how divisors can give rational maps to the projective space and ampleness properties, which are related to divisors that give immersions.

Let L and M be locally free sheaves of rank 1 (i.e. **invertible sheaves**) on a ringed space X, then it is easy to see that:

- $L \otimes_{\mathcal{O}_X} M$ is a locally free sheaf of rank 1;
- $L^{-1} := L^* = \mathrm{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$ is an invertible sheaf and $L \otimes L^* \simeq \mathcal{O}_X$.

Definition 2.1.1. From the comment above, we see that the set of invertible sheaves on a ringed space X has a group structure, with identity given by $\mathcal{O}_X$. This group is called the **Picard group** of X and it is denoted by $\mathrm{Pic}(X)$.
Denote by $\mathrm{CaCl}(X)$ the free abelian group on the set of Cartier divisors on X, with operation given by the sum of divisors and the identity given by the 0 divisor.

The next theorem compares the two groups we just defined. In particular, for smooth varieties they coincide.

Let X be a scheme and let $\mathcal{K}$ be its **sheaf of total quotient rings** of $\mathcal{O}_X$ (i.e. the sheafification of the presheaf $U \mapsto S(U)^{-1}\mathcal{O}_X(U)$, where $S(U)$ is the set of sections of $\mathcal{O}_X(U)$ which are not zero divisors in each local ring $\mathcal{O}_{X,P}$, for all $P \in U$). Let $D = \{(U_i, f_i) | i \in I\}$ be a Cartier divisor on X, where $\{U_i | i \in I\}$ is a cover of X such that D is represented by the functions $f_i \in \mathcal{K}(U_i)$ in each U_i. Define the sheaf $\mathcal{O}_X(D)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on each U_i. Since in $U_i \cap U_j$, $\frac{f_i}{f_j}$ is invertible, f_i^{-1} and f_j^{-1} define the same sheaf in the intersection, thus we get a well defined $\mathcal{O}_X$-module.

Theorem 2.1.2. *If X is an integral scheme (e.g. if X is smooth), there is an isomorphism of groups* CaCl(X) $\to$ Pic(X) *which associates every divisor D to an invertible sheaf denoted by $\mathcal{O}_X(D)$.*

Proof. First of all, note that (keeping the notations above), on each U_i, the map $\mathcal{O}_{U_i} \to \mathcal{O}_X(D)_{|U_i}$ defined by $1 \mapsto f_i^{-1}$ is an isomorphism by construction. So, $\mathcal{O}_X(D)$ is a locally free sheaf of rank 1. Moreover, D can be uniquely recovered from $\mathcal{O}_X(D)$ together with its embedding in $\mathcal{K}$, by taking on each U_i the inverse f_i of a local generator of the sheaf. As $\mathcal{O}_X(D)$ is an invertible subsheaf of $\mathcal{K}$, this gives a unique well defined Cartier divisor on X.
The map $D \to \mathcal{O}_X(D)$ is a group morphism as, if D_1 is locally defined by f and D_2 by g, then $\mathcal{O}_X(D_1 - D_2)$ is locally generated by $f^{-1}g$, so $\mathcal{O}_X(D_1 - D_2) = \mathcal{O}_X(D_1) \cdot \mathcal{O}_X(D_2)^{-1}$ as subsheaves of $\mathcal{K}$ and this product is isomorphic to the tensor product $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^*$.
Now, we show that $D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$, so that the map $D \mapsto \mathcal{O}_X(D)$ descends to a map $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$. As we have already seen that the map is a group morphism, it is enough to show that a principal divisor is associated with the structure sheaf. But this is clear as, if D is principal, it is defined by a meromorphic global section $f \in \mathcal{K}^*$ and $1 \mapsto f^{-1}$ gives a global isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$.
Up to now, we never used the fact that X is an integral scheme. In fact, this property is used only to show that this association is surjective (injectivity already proven). If L is an invertible sheaf together with an embedding in $\mathcal{K}$, then we know how to construct a divisor D whose associated sheaf is L. Thus, it is enough to prove that every invertible sheaf can be embedded as a subsheaf of $\mathcal{K}$. It is a general fact that, if X is an integral

scheme, the sheaf $\mathcal{K}$ is the constant sheaf K, with K the function field of X. Consider the sheaf $L \otimes_{\mathcal{O}_X} \mathcal{K}$. On any open set U where L is isomorphic to $\mathcal{O}_X$, $L \otimes \mathcal{K} \simeq \mathcal{K}$, so it is a constant sheaf on U. But X is irreducible and these opens U cover X, so $L \otimes \mathcal{K}$ must be a constant sheaf on all of X. Thus, $L \otimes \mathcal{K}$ is isomorphic to the constant sheaf $\mathcal{K}$ and the natural map $L \to L \otimes \mathcal{K} \simeq \mathcal{K}$ gives the desired embedding. qed

Remark. From now on we often freely pass from a divisor to the line bundle associated with it.

The goal now is to understand how divisors can give maps of a variety inside a projective space. In particular, we will study the notions of ampleness and of linear systems.

Definition 2.1.3. Let X be a scheme over Y and L an invertible sheaf on it.
An **immersion** is a morphism $i : X \to Z$ that gives an isomorphism with an open subscheme of a closed subscheme of Z. Then, L is said to be **very ample** relative to Y (we will often omit Y when it is clear from the context) if there is an immersion $i : X \to \mathbb{P}^N_Y$ for some N such that $i^*\mathcal{O}(1) \simeq L$.
A sheaf is said to be **generated by global sections** if there exists a subset $\{s_0, ..., s_r\} \subseteq H^0(X, \mathscr{F})$ such that, for each point of X, the images of s_i in the stalks for $i = 0, ..., r$ generate the stalks. If X is a Noetherian scheme, then L is said to be **ample** if, for every coherent sheaf $\mathscr{F}$ on X, there is an integer $n_0 > 0$ such that, for all $m \geq n_0$, $\mathscr{F} \otimes L^{\otimes m}$ is generated by global sections.

Theorem 2.1.4. *Let L be an invertible sheaf on a noetherian scheme of finite type over a ring A, then the followings are equivalent:*

(i) *L is ample;*

(ii) *$L^{\otimes m}$ is ample $\forall m > 0$;*

(iii) *$L^{\otimes m}$ is ample for some $m > 0$;*

(iv) *$L^{\otimes m}$ is very ample for some $m > 0$.*

Proof.

(i) $\Rightarrow$ (ii) Immediate from the definition.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (i) Assume $L^{\otimes m}$ is ample. Let $\mathscr{F}$ be a coherent sheaf on X, then there exists n_0 such that for all $n \geq n_0$, $\mathscr{F} \otimes L^{\otimes mn}$ is generated by global sections. But also $\mathscr{F} \otimes L$ is coherent, so there exists n_1 such that for all $n \geq n_1$, $\mathscr{F} \otimes L^{\otimes(1+mn)}$ is generated by global sections. Repeat this process for the sheaves $\mathscr{F} \otimes L^{\otimes i}$, with $i = 0, ..., m-1$. Then take $N = \max_{i=0,...,m-1} n_i$. Then, for all $n \geq N$, $\mathscr{F} \otimes L^{\otimes n}$ is generated by global sections, hence L is ample.

(iv) ⇒ (iii) $L^{\otimes m}$ is very ample for some m, so there is an immersion $i : X \to \mathbb{P}^n_A$ such that $L^{\otimes m} \simeq i^*\mathcal{O}(1)$. Let $\overline{X}$ be the closure of $i(X)$, it is a projective scheme. Then, a result by Serre ([H, ch.II, theorem 5.17]) tells us that $\mathcal{O}_{\overline{X}}(1)$ is ample on $\overline{X}$. But any coherent sheaf $\mathscr{F}$ on X extends to a coherent sheaf $\overline{\mathscr{F}}$on $\overline{X}$. If $\overline{\mathscr{F}} \otimes \mathcal{O}_{\overline{X}}(1)^{\otimes n}$ is generated by global sections, then also $\mathscr{F} \otimes \mathcal{O}_X^{\otimes n}$ is. Hence, $L^{\otimes m}$ is ample on X.

(i) ⇒ (iv) First of all we want to show that, for every $P \in X$, there exists an $n > 0$ and $s \in H^0(X, L^{\otimes n})$ such that $P \in X_s$ and X_s is affine ($X_s = \left\{Q \in X | s_Q \notin \mathfrak{m}_Q L_Q^{\otimes n}\right\}$). Indeed, let U be an affine open neighborhood of P such that $L_{|U}$ is free, let $Y := X \setminus U$ and $\mathcal{I}_Y$ the sheaf of ideals defining Y. In particular, $\mathcal{I}_Y$ is coherent on X. Thus, for some $n > 0$, $\mathcal{I}_Y \otimes L^{\otimes n}$ is generated by global sections. In particular, there exists $s \in H^0(X, \mathcal{I}_Y \otimes L^{\otimes n})$ such that $s_P \notin \mathfrak{m}_P(\mathcal{I}_Y \otimes L^{\otimes n})_P$. The sheaf $\mathcal{I}_Y \otimes L^{\otimes n}$ is a subsheaf of $L^{\otimes n}$, so we can think of s as a global section of $L^{\otimes n}$. $P \in X_s$ and $X_s \subseteq U$. But $L_{|U}$ is trivial, so $s_{|U} = f \in H^0(U, \mathcal{O}_U)$ and $X_s = U_f$ is affine.
Since X is compact, from the cover $\{X_s\}$ we can extract a finite subcover, say $X_{s_1}, ..., X_{s_k}$. By replacing each s_i with a suitable power, we can assume all $s_i \in L^{\otimes n}$ for the same n (taking powers does not change X_{s_i}). The sheaf $L^{\otimes n}$ is also ample and to show $L^{\otimes m}$ is very ample for some m, we may replace L with $L^{\otimes n}$.
For each $i = 1, ..., k$, let $B_i = H^0(X_{s_i}, \mathcal{O}_{X_{s_i}})$, since X is of finite type, these are finitely generated A-algebras. Call $\{b_{ij} | j = 1, ..., r_i\}$ a set of generators. By [H, lemma 5.14], for each i, j there is an integer $n > 0$ such that $s_i^n b_{i,j}$ extends to a global section $c_{i,j} \in H^0(X, L^{\otimes n})$. Take n large enough to work for all i, j.
Define a morphism

$$\varphi : X \to \mathbb{P}^N_A; \quad x \mapsto [s_1^n(x) : ... : s_k^n(x) : c_{1,1}(x) : ... : c_{k,r_k}(x)]$$

The sections s_i^n generate $L^{\otimes n}$ as X_{s_i} cover X, so this is indeed a morphism. Let $\{x_i, x_{i,j}\}$ be the corresponding coordinates on $\mathbb{P}^N_A$ and let U_i be the open subset where $x_i \neq 0$. Then $\varphi^{-1}(U_i) = X_i$ and the map of affine rings

$$A[x_i, x_{i,j}] \to B_i$$

is surjective. Thus X_i is mapped onto a closed subscheme of U_i and φ gives an immersion of X in a closed subscheme of $\bigcup_{i=1,...,k} U_i \subseteq \mathbb{P}^N_A$. Hence, $L^{\otimes n}$ is very ample.

qed

If X is a smooth projective variety, then the groups of Cartier and Weil divisor coincide and they are isomorphic to the Picard group of the variety. Another useful fact is that, if D is a divisor and $L = \mathcal{O}_X(D)$ is the corresponding element in $\mathrm{Pic}(X)$, the global sections of L describe exactly all the effective divisors which are linearly equivalent to D. Indeed, to each global section s, we can associate its **divisor of zeros** with the following construction. Let $\{U\}$ be a cover of X such that $L_{|U} \simeq \mathcal{O}_{X|U}$ for each U and call φ_U the isomorphisms. The collection $\left\{(U, \varphi_U(s_{|U}))\right\}$ gives a well-defined effective Cartier divisor on X.

Proposition 2.1.5. *Let X be a smooth projective variety over an algebraically closed field k. Let D_0 be a divisor on X and let $L = \mathcal{O}_X(D_0)$ be the corresponding invertible sheaf. Then:*

(i) for each nonzero section $s \in H^0(X, L)$, the divisor of zeros is an effective divisor linearly equivalent to D_0;

(ii) every effective divisor linearly equivalent to D_0 is of the form (s) for a global section $s \in H^0(X, L)$;

(iii) two global sections $s, s' \in H^0(X, L)$ define the same divisor if and only if there exists a scalar $\lambda \in k^*$ such that $s' = \lambda s$.

Proof. (i) Identify L with a subsheaf of $\mathcal{K}$. Then, s corresponds to a meromorphic function f in the function field of X. If D_0 is locally defined by f_i then L is locally generated by f_i^{-1}, so a local isomorphism of L with $\mathcal{O}_X$ is given by multiplication by f_i. Then, D is locally defined as $(f_i f)$. So, $D = D_0 + (f)$, whence $D \sim D_0$.

(ii) Let $D \sim D_0$, $D > 0$. Then, there exists f in the function field of X such that $D = (f) + D_0$; f gives a global section of L.

(iii) Using the same construction as before we get f and f' meromorphic functions on X associated with s and s'. In particular, as $(s) = (s')$, $\left(\frac{f}{f'}\right) = 0$. This means that $\frac{f}{f'} \in H^0(X, \mathcal{O}_X^*) = k^*$ as X is a projective variety over an algebraically closed field. So, $s' = \lambda s$ for a nonzero scalar λ.

qed

Definition 2.1.6. The set of all effective divisors which are linearly equivalent to a given divisor D_0, by the correspondence described above in proposition 2.1.5, is in bijection with $H^0(X, \mathcal{O}_X(D_0))/k^*$, so it has a structure of a projective space (closed points of a projective space). This set is called the **complete linear system associated with** D_0 and is denoted by $|D_0|$ or $|\mathcal{O}_X(D_0)|$.
A **linear system** $\mathfrak{d}$ is a subset of a complete linear system, which corresponds to a linear subspace for the projective structure. The **dimension** of $\mathfrak{d}$ is its dimension as a linear projective variety.

Definition 2.1.7. Let $\mathfrak{d}$ be a linear system on a projective variety X, let F be the maximum divisor such that, for all $D \in \mathfrak{d}$, $D \geq F$. F is called the **fixed divisor of** $\mathfrak{d}$. (We say $D \geq D'$ if $D - D'$ is effective.)
A point $P \in X$ is called a **base point** of $\mathfrak{d}$ if $P \in \text{Supp}(D)$ for all $D \in \mathfrak{d}$. ($\text{Supp}(D)$ is the union of all prime divisors of D.) The set of all base points is called the **base locus** of the linear system and is denoted by $\text{Bs}(\mathfrak{d})$. A linear system is called **base-point-free** if its base locus is empty.

Proposition 2.1.8. *Let $\mathfrak{d}$ be a linear system on a variety X corresponding to a subspace V of the global sections of a line bundle L. Then P is a base point of $\mathfrak{d}$ if and only if $s_P \in \mathfrak{m}_P L_P$ for all $s \in V$. In particular, $\mathfrak{d}$ is base-point-free if and only if L is generated by the global sections in V.*

Proof. The first claim comes directly from the definition of base point noting that a point belongs to the support of a divisor D if and only if it is a zero of the global section defining D.
For the second claim: if V generates L, then, for every point P there exists a global section $s \in V$ such that $s_P \notin \mathfrak{m}_P L_P$. But then, the divisor corresponding to s does not have P in its support. Viceversa, if L was not globally generated by V, there exists P such that all global sections in V belong to $\mathfrak{m}_P L_P$, but this means that P is a base point of $\mathfrak{d}$. qed

Remark. Each linear system $\mathfrak{d}$, with corresponding linear subspace of global sections V with basis $\{s_0, ..., s_N\}$, determines a rational map from the variety X to the projective space $\mathbb{P}(V)$:

$$\Phi_{\mathfrak{d}} : X \dashrightarrow \mathbb{P}(V); \quad x \mapsto [s_0(x) : ... : s_N(x)].$$

Note that $\Phi_{\mathfrak{d}}$ is well-defined outside the base locus of $\mathfrak{d}$. Moreover, if F is the fixed divisor of $\mathfrak{d}$, then the rational map induced by $\mathfrak{d} - F$ is still $\Phi_{\mathfrak{d}}$.

Remark. Let $\mathfrak{d}$ be a linear system and $\mathfrak{f}$ a subsystem of it on a projective variety X. Let V be the vector space associated with $\mathfrak{d}$ and W the subspace determined by $\mathfrak{f}$. Note that there is an inclusion of the base loci: $\mathrm{Bs}(\mathfrak{d}) \subseteq \mathrm{Bs}(\mathfrak{f})$, so both $\Phi_{\mathfrak{d}}$ and $\Phi_{\mathfrak{f}}$ are well-defined on $X \setminus \mathrm{Bs}(\mathfrak{f})$. (Note that, if $\mathfrak{f}$ is base-point-free, also $\mathfrak{d}$ is.) The map $\Phi_{\mathfrak{f}}$ can be seen as the composition of $\Phi_{\mathfrak{d}}$ and the projection on the appropriate components of the projective space target of $\Phi_{\mathfrak{d}}$. But this projection, on the closure of the image of X, Y, is finite (as Y is projective, see [G, lemma 4.1.5 and remark 4.1.6]).

Remark. A very ample divisor L on a variety X is characterized by the fact that there exists an immersion $i : X \to \mathbb{P}^n$ such that $L \simeq i^*\mathcal{O}(1)$. This is equivalent to saying that there exists a subset of global sections $\{s_0, ..., s_n\}$ that generates L and such that the corresponding linear system gives an immersion.
In particular, if L is ample, a power of it is very ample, say $L^{\otimes m}$, so the linear system $|L^{\otimes m}|$ gives an immersion of X in a projective space.

2.1.2 Canonical sheaf

Here we see the properties of a particular divisor of smooth varieties: the canonical divisor. It is defined as the largest exterior power of the sheaf of differentials and it carries a lot of information about the variety. For example we saw that for curves the sign of its degree determines whether C is potentially dense or not. In higher dimension, conjecturally, positivity of the canonical bundle again governs potential density of a variety.

Definition 2.1.9. Let X be a smooth variety over a field k. Then its sheaf of differentials $\Omega_{X/k}$ (or simply Ω_X) is a locally free sheaf of rank n equal to the dimension of X ([H, ch.II, theorem 8.15]). Its n^{th} exterior power is a locally free sheaf of rank 1 and it is called **canonical sheaf** of X and denoted by ω_X. The corresponding divisor is called the **canonical divisor** of X and is denoted by K_X.

Example 2.1.10. Let k be a field and $X = \mathbb{P}^n_k$. We want to compute ω_X. There is an exact sequence ([H, ch.II, theorem 8.13]):

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0.$$

Then, taking the largest exterior power, we get that $\omega_X = \mathcal{O}_X(-n-1)$

Next, we study how to compute the canonical sheaf of a subvariety.

Definition 2.1.11. Let Y be a smooth subvariety of a smooth variety X over a field k. Let $\mathcal{I}$ be the ideal subsheaf of $\mathcal{O}_X$ defining Y. The locally free sheaf $\mathcal{I}/\mathcal{I}^2$ is called the **conormal sheaf** of Y in X and its dual $\mathcal{N}_{X/Y} = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is called the **normal sheaf** of Y in X. It is locally free of rank $r = \mathrm{codim}_X(Y)$, the codimension of Y in X.

Proposition 2.1.12. *Adjunction formula*
Let Y be a smooth subvariety of codimension r in a smooth variety X over a field k. Then:

$$\omega_Y \simeq \omega_X \otimes \Lambda^r \mathcal{N}_{Y/X}.$$

In the case $r = 1$, consider Y as a divisor and call L its corresponding invertible sheaf. Then:

$$\omega_Y \simeq \omega_X \otimes L \otimes \mathcal{O}_Y.$$

Proof. Let $\mathcal{I}$ be the ideal sheaf defining Y. Then, there is an exact sequence of sheaves on Y ([H, ch.II, theorem 8.17]):

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \to 0.$$

Take the largest exterior powers to get $\omega_X \otimes \mathcal{O}_Y \simeq \omega_Y \otimes \Lambda^r(\mathcal{I}/\mathcal{I}^2)$. Tensor the above equality with $\Lambda^r \mathcal{N}_{Y/X}$, which is the dual of $\Lambda^r(\mathcal{I}/\mathcal{I}^2)$:

$$\omega_X \otimes \Lambda^r \mathcal{N}_{Y/X} \simeq \omega_Y \otimes \Lambda^r(\mathcal{I}/\mathcal{I}^2) \otimes \Lambda^r \mathcal{N}_{Y/X} \simeq \omega_Y.$$

In the case where $r = 1$, note that, by construction, $L^{-1} \simeq \mathcal{I}$. Thus, $\mathcal{I}/\mathcal{I}^2 \simeq L^{-1} \otimes \mathcal{O}_Y$ and so its dual $\mathcal{N}_{Y/X} \simeq L \otimes \mathcal{O}_Y$. The above result applied to this situation gives the formula. qed

Example 2.1.13. Let H_d be an hypersurface of degree d in $\mathbb{P}^n_k$, for a field k. Then, H_d is defined as the zero locus of a homogeneous polynomial of degree d, namely $p(x_0, ..., x_n)$. The function $f = \frac{p}{x_0^d}$ defines a linear equivalence between H_d and the divisor dH (where

H is the zero locus of x_0). Thus, the sheaf associated with H_d is exactly $\mathcal{O}_X(H_d) = \mathcal{O}_X(d)$. From the adjunction formula above 2.1.12 and the example 2.1.10, we can then compute

$$\omega_{H_d} = \omega_{\mathbb{P}^n_k} \otimes \mathcal{O}_X(H_d) \otimes \mathcal{O}_{H_d} = \mathcal{O}_{H_d}(d-n-1).$$

Proposition 2.1.14. *[SP, lemma 29.31.9 and 29.32.16] Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Then, there is an exact sequence of sheaves on X:*

$$f^*\Omega_{Y/X} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Moreover, if f is smooth, the sequence is exact also on the left.

2.2 Iitaka and Kodaira dimension

In dimension one, the genus of a curve ($g = h^0(C, K_C)$) carries a lot of information about the curve itself. This notion in higher dimensions is more complicated and is generalized using the asymptotic behavior of linear systems associated with the sheaves $K_X^{\otimes m}$. Actually, we can study this behavior for general line bundles (Iitaka dimension) and then specialize the definition to the canonical bundle.

Given a projective variety X and a line bundle L, let

$$N(X, L) := \{m \in \mathbb{N} | H^0(X, L^{\otimes m}) \neq 0\}.$$

Note that, if $m \in N(X, L)$, then all its multiples are in the set, as if $s \in H^0(X, L^{\otimes m})$ is a nonzero section, then $s \otimes ... \otimes s$ (c times) belongs to $H^0(X, L^{\otimes mc})$ and it is a nonzero section. Define the **exponent** of L to be $e = \gcd(m \in N(X, L))$. All large enough multiples of the exponent are in fact in $N(X, L)$. Indeed, let $m \in N(X, L)$, then $m = e\mu$ for an integer μ and all multiples of m are in $N(X, L)$. But for every prime p dividing μ, there must exist $m_p \in N(X, L)$ such that $m_p = e\mu_p$ and $\gcd(\mu_p, p) = 1$. All multiples of m_p belong to $N(X, L)$. But then $\gcd(\mu, \{\mu_p | \, p|\mu\}) = 1$, so every large enough integer can be written as a combination with natural coefficients of μ and the μ_p's. But this implies that all large enough multiples of e belong to $N(X, L)$.

Definition 2.2.1. Let X be a projective variety and L a line bundle on it. Consider $\nu : \hat{X} \to X$ its normalization with the line bundle ν^*L. For each $m \in N(\hat{X}, \nu^*L)$, consider the rational map defined by the linear system $|\nu^*L^{\otimes m}|$ and call Y_m the closure of its image.

$$\Phi_m := \Phi_{|\nu^*L^{\otimes m}|} : \hat{X} \dashrightarrow Y_m \subseteq \mathbb{P}(H^0(\hat{X}, \nu^*L^{\otimes m}))$$

The **Iitaka dimension** of (X, L) is

$$\kappa(X, L) = \max_{m \in N(\hat{X}, \nu^*L)} \{\dim Y_m\}.$$

If $N(\hat{X}, \nu^*L)$ is empty, define $\kappa(X, L)$ to be $-\infty$.
In particular, $\kappa(X, L) = -\infty$ or $0 \leq \kappa(X, L) \leq \dim X$.

Definition 2.2.2. If X is smooth, let K_X be its canonical divisor, then the **Kodaira dimension** of X is $\kappa(X) := \kappa(X, K_X)$. If X is not smooth, its Kodaira dimension is defined as the Kodaira dimension of a desingularization of it.

Remark. In the definitions above we need to consider the normalization of the variety (or a desingularization) to have a birational invariant. If we do not take it into account we may get a different result which is not preserved by birational equivalence (see [L1, examples 2.1.6 and 2.1.8]).

Example 2.2.3.

- If L correponds to a divisor $-D$, with D effective, then $H^0(X, -mD)$ is empty for all m, so $\kappa(X, L) = -\infty$.
- $\kappa(X, L) = 0$ if and only if $h^0(X, L^{\otimes m}) \leq 1$, for all $m \in N(X, L)$ and there exists an m such that equality holds. Indeed, if $h^0(X, L^{\otimes m}) = 1$, the induced map has image of dimension zero, a point.
- If $h^0(X, L^{\otimes m}) \leq 1$ and L is torsion in $\mathrm{Pic}(X)$, then $\kappa(X, L) = 0$. Indeed, $L^{\otimes m} = \mathcal{O}_X$ for some m and $\mathcal{O}_X(X) = k$, the base field of X, as X is a projective variety.
- $\kappa(X, L) = \dim X$ if and only if for some $m \in N(X, L)$, $L^{\otimes m}$ is **big**, which means that $L^{\otimes m} = A + E$ for an ample divisor A and an effective divisor E. This is a consequence of the following lemma:

  Lemma 2.2.4. *[L1, corollary 2.2.7] Let D be a divisor on X. Then, D is big if and only if there exists m such that $\Phi_{|mD|}$ is birational onto its image.*

- If X is a curve, denote by g its genus. Then, we have $\kappa(X) = -\infty$ if $g = 0$, $\kappa(X) = 0$ if $g = 1$ and $\kappa(X) = 1$ if $g \geq 2$.

Next, we see that the Kodaira dimension is preserved by fibrations. Before that, we need to study some general results about fibrations.

Definition 2.2.5. A (regular) map $f : X \to Y$ between projective reduced and irreducible varieties is called a **fibration** if it is surjective with connected fibers.

Theorem 2.2.6. *Stein factorization [H, ch.III, corollary 11.5]*
Let $f : X \to Z$ be a projective morphism of noetherian schemes. Then $f = g \circ f'$, where $f' : X \to Y$ is a projective morphism with connected fibers and $g : Y \to Z$ is a finite morphism.

Lemma 2.2.7. *Let $f : X \to Y$ be a surjective morphism between projective normal varieties over an algebraically closed field k, then the followings are equivalent:*

(a) f is a fibration;

(b) the finite part in the Stein factorization of f is trivial;

(c) $f_*\mathcal{O}_X = \mathcal{O}_Y$*;*

(d) if $k(Y) \subseteq k(X)$ *is the field extension induced by* f *on the function fields,* $k(Y)$ *is algebraically closed in* $k(X)$*.*

Proof.

(a) $\Leftrightarrow$ (b) This comes directly from the definitions.

(a) $\Leftarrow$ (c) See [H, ch.III, corollary 11.3].

(b) $\Rightarrow$ (c) This comes directly from the construction of the Stein factorization, see [H, ch.III, corollary 11.5].

(a) $\Rightarrow$ (d) If $k(Y)$ was not algebraically closed in $k(X)$, then there is a non-trivial finite extension of $k(Y)$ inside $k(X)$, which corresponds to a factorization of f as $X \xrightarrow{u} Y' \xrightarrow{v} Y$, where v is finite of degree ≥ 2, so its fibers are not connected.

(d) $\Rightarrow$ (b) If $k(Y)$ is algebraically closed in $k(X)$, then the finite part of the Stein factorization must be trivial.

qed

Proposition 2.2.8. *[B, theorem 7.1] Let* $f : X \to Y$ *be a fibration from an irreducible non-singular algebraic variety* X *such that the induced field extension* $k(Y) \subseteq k(X)$ *is separable. Then, the general fiber of* f *is geometrically integral. (Recall that* ***geometrically integral*** *means that it is integral for all extensions of the base field. Recall also that a variety is integral if and only if it is irreducible and reduced.)*

Proposition 2.2.9. *If* $p : X \to Y$ *is a fibration, then* $\kappa(X, p^*L) = \kappa(Y, L)$ *for any line bundle* L *on* Y*.*

Proof. Generally, $H^0(X, p^*L) = H^0(Y, p_*p^*L)$. But, by lemma 2.2.7, $p_*\mathcal{O}_X = \mathcal{O}_Y$, so $p_*p^*L = p^*\mathcal{O}_X \otimes L = L$, so:

$$H^0(X, p^*L) = H^0(Y, L).$$

Without loss of generality, we can assume X and Y are smooth. Let $\Phi_{|L^{\otimes m}|} : Y \dashrightarrow \mathbb{P}^N_k$ be the rational map defined by the linear system $|L^{\otimes m}|$, then $\Phi_{|L^{\otimes m}|} \circ p$ is the map defined by the linear system $|p^*L^{\otimes m}|$, by the equality proven above. Viceversa, the map defined by the linear system $|p^*L^{\otimes m}|$ factorizes through p and $\Phi_{|L^{\otimes m}|}$, again by the equality proven above. qed

Lemma 2.2.10. *If* $f : X \to Y$ *is a fibration between projective irreducible varieties, then the induced morphism on the Picard groups is injective.*

Proof. If B is a line bundle on Y such that $f^*B \simeq \mathcal{O}_X$, then $H^0(Y, B) = H^0(X, f^*B) \neq 0$ by the previous proposition 2.2.9. Similarly, $H^0(Y, B^{-1}) = H^0(X, f^*B^{-1}) \neq 0$. Let $0 \neq s \in H^0(Y, B)$ and $0 \neq \sigma \in H^0(Y, B^{-1})$, then $s \otimes \sigma \in H^0(Y, B \otimes B^{-1})$. Thus $s \otimes \sigma$ is a constant. Moreover, it is non-zero since the fact that both s and σ are non-zero implies that there exists an open subset of X where they both do not vanish anywhere. Hence, $s \otimes \sigma$ is a constant which does not vanish anywhere, whence we deduce that s and σ do not vanish anywhere. Therefore they are constants and $B = \mathcal{O}_Y$. qed

Lemma 2.2.11. *[D, lemma 1.15] Let X, Y, Y' be three varieties related by two proper morphisms $f : X \to Y$, $f' : X \to Y'$ such that $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ (i.e. it is a fibration) and such that f' contracts the fibers of f. Then, f' factors through f.*

The following propositions give the first properties of the Kodaira dimension of a variety.

Proposition 2.2.12. *Let X be a smooth projective variety, then $\kappa(X)$ is preserved by birational maps.*

Proof. Let $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ be two birational maps inverse to each other. Let V be the largest subset of X where φ is well-defined. Then, from 2.1.14 there is an induced map $\varphi^*\Omega_Y \to \Omega_V$. They are locally free sheaves of the same rank, $\dim X$, so we get an induced map on the canonical bundles $\varphi^*\omega_Y \to \omega_V$, which gives a map on the global sections. As φ is birational, there is an open subset $U \subseteq V$ such that $\varphi(U)$ is open in Y. Thus, $\omega_{V|U} \simeq \omega_{X'|\varphi(U)}$ via φ. As a nonzero global section cannot vanish on a dense open subset, the map of vector spaces $\varphi^* : H^0(X', \omega_{X'}) \to H^0(V, \omega_V)$ must be injective.
Now, we want to compare $H^0(V, \omega_V)$ and $H^0(X, \omega_X)$. We claim that, as a consequence of the valuatative criterion of properness 1.1.3, $X \setminus V$ has codimension at least 2. Indeed, if H is a point of codimension 1 in X then $\mathcal{O}_{X,H}$ is a discrete valuation ring (as X smooth). Consider the diagram below, where η are the generic points.

$$\begin{tikzcd} X_\eta \arrow[r, "\varphi"] \arrow[d] & Y \arrow[d, "f"] \\ Spec(\mathcal{O}_{X,H}) \arrow[r] \arrow[ur, dashed, "\exists ! \widetilde{H}"] & Spec(k) \end{tikzcd}$$

By the valuatative criterion of properness for Y we get a unique $\widetilde{H}$ compatible with φ, but this means we can extend φ in a neighborhood of H, because $\mathcal{O}_{X,H}$ is a direct limit, thus every function defined on it is defined in a neighborhood of the point H. So, $H \in V$ by definition of V.
We claim that the natural restriction $H^0(X, \omega_X) \to H^0(V, \omega_V)$ is bijective. For any U affine and small enough, $\omega_{X|U} \simeq \mathcal{O}_U$, so $H^0(U, \omega_U) \to H^0(U \cap V, \omega_{U \cap V})$ is bijective as $U \setminus U \cap V$ has codimension at least 2 in U. So the height one primes in U coincide with the height one primes in $U \cap V$. The result is then a consequence of the fact that, if A is an integrally closed noetherian domain, then $A = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$, where the intersection is taken

over all height one prime ideals. As these opens U cover X, we have the claim.
We proved we have an inclusion $\varphi^* : H^0(X', \omega_{X'}) \to H^0(V, \omega_V) \simeq H^0(X, \omega_X)$; using ψ we get the reverse inclusion, whence the result. qed

Definition 2.2.13. Let $f : X \to Y$ be a morphism of varieties, then f is called **finite étale cover** if it is smooth of relative dimension 0. i.e.

(a) f is flat;

(b) if $X' \subseteq X$ and $Y' \subseteq Y$ are irreducible components such that $f(X') \subseteq Y'$, then $\dim X' = \dim Y'$;

(c) for each point $x \in X$ (closed or not), $\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = 0$.

Proposition 2.2.14. *Let X be a smooth projective variety, then $\kappa(X)$ is preserved by finite étale covers.*

Proof. Let $f : X \to Y$ be a finite étale cover with X and Y smooth. Then, as stated in 2.1.14, we have an exact sequence:

$$f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0.$$

But, as f is finite étale, $\Omega_{X/Y} = 0$, so $f^*\Omega_{Y/k} \to \Omega_{X/k}$ is surjective. But $\dim X = \dim Y := n$ and the local rank of $f^*\Omega_{Y/k}$ and $\Omega_{X/k}$ is exactly n (by smoothness). Since the induced local map is a morphism of k-vector spaces, it must be injective as well. Hence, $f^*\Omega_{Y/k} \simeq \Omega_{X/k}$, so $f^*\omega_Y \simeq \omega_X$. Therefore, we can apply proposition 2.2.9 to get $\kappa(X) = \kappa(Y)$. qed

Proposition 2.2.15. *Additivity for products*
Let Y and Z be smooth projective varieties, $X := Y \times Z$, then $\kappa(X) = \kappa(Y) + \kappa(Z)$.

Proof. Denote by p_1 and p_2 the two projections from X. Locally, in some affine subsets, we can describe Y and Z respectively as $\operatorname{Spec}\left(\frac{k[y_1,...,y_r]}{I(Y)}\right)$ and $\operatorname{Spec}\left(\frac{k[z_1,...,z_s]}{I(Z)}\right)$ for some r, s and some ideals $I(Y)$ and $I(Z)$. Then in the corresponding affine in X, the product variety is described as

$$\operatorname{Spec}\left(\frac{k[y_1,...,y_r]}{I(Y)} \otimes \frac{k[z_1,...,z_s]}{I(Z)}\right) \simeq \operatorname{Spec}\left(\frac{k[y_1,...,y_r,z_1,...,z_s]}{I(Y), I(Z)}\right)$$

where in the last equality $I(Y)$ and $I(Z)$ are extended to ideals in $k[y_1,...,y_r,z_1,...,z_s]$. So,

$$\Omega_X \simeq \langle dy_1,...,dz_1,...,dz_s | \text{differentials of polynomials in } I(Y), I(Z) = 0\rangle \simeq p_1^*\Omega_Y \oplus p_2^*\Omega_Z$$

Then, taking exterior powers, we get $\omega_X = p_1^*\omega_Y \otimes p_2^*\omega_Z$. So, if $\{s_i | i = 0,...,d\}$ is a set of generators for the global sections of $\omega_Y^{\otimes m}$ and $\{t_j | j = 0,...,d'\}$ is a set of generators for $\omega_Z^{\otimes m}$, then the set $\{\sigma_{ij} = (s_i \circ p_1) \otimes (t_j \circ p_2)\}$ generates the global sections of $\omega_X^{\otimes m}$. Then, the map defined by $\omega_X^{\otimes m}$ can be described as

$$x \mapsto [s_0(p_1(x))t_0(p_2(x)) : ... : s_d(p_1(x))t_{d'}(p_2(x))].$$

Without loss of generality, we may assume $s_0, t_0 \neq 0$, so that we can recover each s_i and t_j as $\frac{\sigma_{i0}}{t_0}$ and $\frac{\sigma_{0j}}{s_0}$. Moreover, note that if we know all s_i and t_j, we have completely determined also all σ_{ij}. So the map

$$f : [\sigma_{00}(x) : ... : \sigma_{dd'}(x)] \mapsto ([\sigma_{10}(x) : ... : \sigma_{d0}(x)], [\sigma_{01}(x) : ... : \sigma_{0d'}(x)])$$

is an isomorphism. Furthermore, composing f with the two projections, we get exactly the two maps defined by $p_1^*\omega_Y^{\otimes m}$ and $p_2^*\omega_Z^{\otimes m}$, which factor through Y and Z. Thus, the dimension of the image of $\Phi_{|\omega_X^{\otimes m}|}$ is exactly $\dim \operatorname{im} \Phi_{|\omega_Y^{\otimes m}|} + \dim \operatorname{im} \Phi_{|\omega_Z^{\otimes m}|}$. But, for m big enough, these dimensions stabilize to the Kodaira dimensions of the varieties (we will see it later, proposition 2.4.4), whence the claim. qed

Example 2.2.16. *Hypersurfaces in* $\mathbb{P}_k^n$
Let H_d be an hypersurface of $\mathbb{P}_k^n$, then by the example 2.1.13, we know that $\omega_{H_d} = \mathcal{O}_{H_d}(d-n-1)$. So:

- if $d < n+1$, there are no global sections and $\kappa(H_d) = -\infty$;
- if $d = n+1$, the only global sections are the constants and $\kappa(H_d) = 0$;
- if $d > n+1$, the canonical divisor is ample, so a power of it gives an immersion, thus $\kappa(H_d) = n-1$.

In the next example (but also later), we will use two embeddings that we wish to recall: the Veronese and the Segre embeddings. Let $M_1, ..., M_N$ be all the monomials of degree d in $x_0, ..., x_n$, the $d^{\mathbf{th}}$ **Veronese embedding** of $\mathbb{P}_k^n$ is defined as:

$$v_d : \mathbb{P}_k^n \to \mathbb{P}_k^N; \quad [x_0 : ... : x_n] \mapsto [M_1 : ... : M_N].$$

Let $z_{i,j} := x_i y_j$, the **Segre embedding** of the product $\mathbb{P}_k^n \times \mathbb{P}_k^m$ is defined as:

$$\mathbb{P}_k^n \times \mathbb{P}_k^m \to \mathbb{P}_k^N; \quad ([x_0 : ... : x_n], [y_0 : ... : y_m]) \mapsto [z_{0,0} : z_{0,1} : ... : z_{n,m}].$$

Example 2.2.17. *Hypersurfaces in* $\mathbb{P}_k^{n-j} \times \mathbb{P}_k^j$
Let $H := H_{d,d'}$ be an hypersurface of bidegree (d, d') in $\mathbb{P}_k^{n-j} \times \mathbb{P}_k^j$. Choose a coordinate system for the two projective spaces and call $x_0, ..., x_{n-j}$ and $y_0, ..., y_j$ the coordinate functions on the two spaces respectively. Then, H is defined by a polynomial $p(x_0, ..., y_0, ..., y_j)$ bihomogeneous of degree d in x and d' in y. The function $f = \frac{p}{x_0^d y_0^{d'}}$ gives a linear equivalence between H and $dH_1 + d'H_2$, where $H_1 = V(x_0)$ and $H_2 = V(y_0)$ as it descends to a meromorphic function $f = \frac{p x_0^{d'-d}}{x_0^{d'} y_0^{d'}}$ or $f = \frac{p y_0^{d-d'}}{x_0^d y_0^d}$ on the Segre embedding of $\mathbb{P}_k^{n-j} \times \mathbb{P}_k^j$. Consider the two projections p_1 and p_2 and

call J_1 and J_2 two hypersufaces of degree 1 in $\mathbb{P}_k^{n-j}$ and $\mathbb{P}_k^j$ respectively. Then $\omega_{H_1} = p_1^*\mathcal{O}_{J_1}(d-n-j-1) = \mathcal{O}_{H_1}(d-n-j-1)$ and $\omega_{H_2} = p_2^*\mathcal{O}_{J_2}(d'-j-1) = \mathcal{O}_{H_2}(d'-j-1)$. Thus, $\omega_H = \mathcal{O}_{\mathbb{P}_k^{n-j}\times\mathbb{P}_k^j}(d-n-j-1, d'-j-1)_{|H}$. Let $a = d-n-j-1$ and $b = d'-j-1$, so that $\mathcal{O}_{\mathbb{P}_k^{n-j}\times\mathbb{P}_k^j}(d-n-j-1, d'-j-1) = \mathcal{O}_{\mathbb{P}_k^{n-j}\times\mathbb{P}_k^j}(aH_1+bH_2)$.

- If $d < n+j+1$ or $d' < j+1$, then there are no global sections, so $\kappa(H) = -\infty$.
- If $d > n+j+1$ and $d' > j+1$, then we can apply to $\mathbb{P}_k^{n-j}\times\mathbb{P}_k^j$ first the two Veronese embeddings of degree a and b, (v_a, v_b) and then the Segre embedding s. So, $\mathcal{O}_{\mathbb{P}_k^{n-j}\times\mathbb{P}_k^j}(aH_1+bH_2) = (s(v_a,v_b))^*\mathcal{O}_{\mathbb{P}_k^N}(1)$ for an N, so it is ample. Thus, a power of it gives an immersion and $\kappa(H) = n-1$.
- If $d = n+j+1$ and $d' > j+1$, consider the maps Φ_m determined by $|\mathcal{O}_{\mathbb{P}_k^{n-j}\times\mathbb{P}_k^j}(0,b)|$ and its powers. In the first component it is always constant, while it gives an immersion in the second component. We can compute the Kodaira dimension of H over an algebraic closure of the field of definition, so we may assume k is algebraically closed. For every $[y_0 : ... : y_j]$, by Nullstellensatz, there exists a point $[x_0 : ... : x_{n-j}]$ such that $p(x_0, ..., y_0, ..., y_j) = 0$. So the image of $\Phi_{m|H}$ coincides with the image of Φ_m. Thus, $\kappa(H) = j$.
- If $d > n+j+1$ and $d' = j+1$, with the same steps, we find $\kappa(H) = n-j$.

2.3 Easy additivity

In this section we see a result which relates the Iitaka dimension of a variety to the Iitaka dimension of fibers in a fibration. One of the two inequalities is still a conjecture in many cases.

For the next theorem, we need to construct a projective space associated with a coherent sheaf. This is not done in the category of schemes, but in the category of analytic spaces. We only sketch its construction here, for more details, see [U, examples 2.8 and 2.9]. Let $\underline{Sets}$ be the category of sets and $\underline{An}(X)$ the category of analytic spaces over a complex space X (varieties over a number field can be interpreted as so with a base field extension). Let $\mathscr{F}$ be a coherent sheaf on X, define a contravariant functor:

$$\mathbb{P}_{\mathscr{F}} : \underline{An}(X) \to \underline{Sets}$$
$$(f : Y \to X) \mapsto \text{set of invertible sheaves on Y which are quotients of } f^*\mathscr{F}.$$

This functor is representable, call $\mathbb{P}(\mathscr{F}) \to X$ its representative in $\underline{An}(X)$, it is called the **projective fiber space associated with** $\mathscr{F}$. If $\mathscr{F} \simeq \mathcal{O}_X^{N+1}$ is a free sheaf of rank $N+1$, then $\mathbb{P}(\mathscr{F})$ is simply $X \times \mathbb{P}^N \to X$, the first projection. If $\mathscr{F}$ is locally free of rank $N+1$, then the projective fiber space is constructed locally using the first projection as above.

Theorem 2.3.1. *Easy additivity*
Let $p : X \to Z$ be a fibration and $L \in \mathrm{Pic}(X)$. Then, for a general fiber X_z of p, the following formula holds:

$$\kappa(X, L) \leq \kappa(X_z, L_{|X_z}) + \dim Z$$

Proof. Let $F_m = p_* L^{\otimes m}$. There exists U_m an open dense subset of Z such that $F_{m|U_m}$ is locally free and for every $u \in U_m$, $X_u := p^{-1}(u)$ is a non-singular fiber (see [U, corollary 1.8] for details). If $W \subseteq U_m$ is an open subset such that $F_{m|W} \simeq \mathcal{O}^{q+1}_{Z|W}$, let $\{\widetilde{\varphi}_0, ..., \widetilde{\varphi}_q\}$ be a set of local generators so that $L^{\otimes m}_{|p^{-1}(W)}$ is locally generated by $\{\varphi_0, ..., \varphi_q\}$, with $\widetilde{\varphi}_i \circ p = \varphi_i$. Define locally:

$$h^{(m)} : p^{-1}(W) \to W \times \mathbb{P}^q; \quad x \mapsto (p(x), [\varphi_0(x) : ... : \varphi_q(x)]);$$
$$g^{(m)} : W \times \mathbb{P}^q \to W; \quad (z, y) \mapsto z.$$

This maps can be glued together (in the category of analytic spaces) to obtain:

$$\begin{tikzcd} X \arrow[r, dashed, "h^{(m)}"] \arrow[dr, "p"'] & \mathbb{P}(F_m) \arrow[d, "g^{(m)}"] \\ & Z \end{tikzcd}$$

where $\mathbb{P}(F_m)$ is the projective fiber space associated with F_m. By Grauert's proper mapping theorem, for every $y \in Z$, there exists U_y open dense subset of Z and $\psi_0, ..., \psi_r \in H^0(p^{-1}(U_y), L^{\otimes m})$ such that for every $z \in U_y$, if we call X_z the fiber above z, $H^0(X_z, L^{\otimes m}_{|X_z})$ is generated by $\psi_0, ..., \psi_r$. Thus,

$$h^{(m)}_{|X_y} = \Phi_{|L^{\otimes m}|, |X_y} : X_y \dashrightarrow \mathbb{P}(F_m)_y; \quad x \mapsto (y, [\psi_0(x) : ... : \psi_q(x)]).$$

But, also, there exists an open dense subset $U \subseteq U^{(m)}$ such that $\dim_{\mathbb{C}} H^0(X_y, L^{\otimes m}_{|X_y})$ is constant for all $y \in U$. Then,

$$\dim h^{(m)}(X) = \dim Z + \dim \Phi_{|L^{\otimes m}_{|X_y}|}(X_y) \leq \dim Z + \kappa(X_y, L_{|X_y})$$

for all $y \in U$, since $\Phi_{|L^{\otimes m}|, |X_y}(X_y) = \mathbb{P}(F_m)_y$, the fiber of $g^{(m)}$ over y. Next, we want to define locally a map $h : \mathbb{P}(F_m) \to \mathbb{P}^q$. If $H^0(X, L^{\otimes m})$ is generated by $\{\varphi_0, ..., \varphi_N\}$, then $H^0(Z, p^* L^{\otimes m})$ is generated by $\{\widetilde{\varphi}_0, ..., \widetilde{\varphi}_N\}$, where $\varphi_i = \widetilde{\varphi}_i \circ p$. Define h locally as:

$$h : W \times \mathbb{P}^q \to \mathbb{P}^N; \quad (u, y) \mapsto [\widetilde{\varphi}_0(u) : ... : \widetilde{\varphi}_N(u)],$$

so that $h \circ h^{(m)} = \Phi_{|L^{\otimes m}|}$. Then,

$$\dim \Phi_{|L^{\otimes m}|}(X) = \dim h \circ h^{(m)}(X) \leq \dim h^{(m)}(X) \leq \dim Z + \kappa(X_y, L_{|X_y})$$

for every $y \in U$. Taking $m \gg 0$ gives the result (for $m \gg 0$ the dimension stabilizes to the Kodaira dimension, we will see this in proposition 2.4.4 below). qed

Theorem 2.3.2. *[15]*
In the above setting, taking L to be the canonical bundle, if Z is of ***general type*** *(i.e. $\kappa(Z) = \dim Z$), then equality holds:*

$$\kappa(X) = \kappa(X_z) + \dim Z.$$

Conjecture 2.3.3. *$C_{n,m}$*[1] *conjecture*
In the above setting, it is conjectured that:

$$\kappa(X) \geq \kappa(X_z) + \kappa(Z).$$

2.4 Iitaka–Moishezon fibration

For m big enough the maps $\Phi_{|L^{\otimes m}|}$ stabilize and they are all birationally equivalent to a fixed well-defined fibration, the **Iitaka–Moishezon fibration** (called also **Iitaka fibration**). We start proving this result for base-point-free linear systems, then we prove the general case.

Before starting the discussion, we need some preliminary results about normalization of varieties and resolutions.

Definition 2.4.1. [H, ch.I, exercise 3.17]
Let X be a variety, then there exists a normal variety $\widetilde{X} \to X$ such that, for every $f : Y \to X$ dominant with Y normal, there exists a unique $\widetilde{f} : Y \to \widetilde{X}$ making the following diagram commutative:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "\widetilde{f}"'] & X \\ & \widetilde{X} \arrow[u] \end{tikzcd}$$

Such $\widetilde{X}$ is called the **normalization** of X.

Remark. $\widetilde{X}$ is constructed affine locally glueing together the spectrum of the normal closure of the coordinate rings.

Lemma 2.4.2. *Let $f : X \to Y$ be a fibration with X normal, then also Y is normal.*

Proof. Let $\nu : \widetilde{Y} \to Y$ be a normalization of Y, then, by the universal property of the normalization, f factors through ν, but ν is a finite map, so this forces ν to be an isomorphism. qed

Theorem 2.4.3. *Hironaka's theorem [12, ch.0.5, question E and main theorem II]*
Let $f : X \dashrightarrow Y$ be a rational map. Then f can be resolved into a regular map by a sequence of blow-ups with smooth centers: $\pi : X_{(m)} \to \ldots \to X_{(0)} = X$ such that $f \circ \pi$ is regular. In particular, we can choose $X_{(m)}$ to be normal.

[1] The name $C_{n,m}$ comes from "contractions", which is another way to call fibrations, from a variety of dimension n to one of dimension m.

A first result on the asymptotic behavior of Iitaka dimension says that this dimension is actually achieved by all sufficiently large $m \in N(X, L)$.

Proposition 2.4.4. *Let X be a normal projective variety and L a line bundle on it, if $\kappa(X, L) = \kappa$, then $\dim \Phi_{|L^{\otimes m}|}(X) = \kappa$ for all sufficiently large $m \in N(X, L)$.*

Proof. Denote by $\Phi_k = \Phi_{|L^{\otimes k}|}$. If $k = -\infty$, the statement is obvious, so let us assume $k \geq 0$. Let e be the exponent of L, possibly by replacing L with $L^{\otimes e}$, we may assume $e = 1$. There exists $p_0 > 0$ such that $H^0(X, L^{\otimes p}) \neq 0$ for every $p \geq p_0$. Fix k such that $\dim \Phi_k(X) = \kappa$. Multiplying by a non-zero section in $H^0(X, L^{\otimes p})$, for every $p \geq p_0$, we have an embedding: $H^0(X, L^{\otimes k}) \subseteq H^0(X, L^{\otimes k+p})$. Let $\{s_0, ..., s_r\}$ be a set of generators for $H^0(X, L^{\otimes k})$ and complete it to a set of generators for $H^0(X, L^{\otimes k+p})$, $\{s_0, ..., s_t\}$. Then, calling ν_p the projection on the first $r + 1$ components, we have a factorization:

$$\Phi_k : X \overset{\Phi_{k+p}}{\dashrightarrow} \mathbb{P}^t \overset{\nu_k}{\dashrightarrow} \mathbb{P}^r; \quad x \mapsto [s_0(x) : ... : s_t(x)] \mapsto [s_0(x) : ... : s_r(x)].$$

Therefore $\dim \Phi_{k+p}(X) \geq \Phi_k(X) = \kappa$, so, by maximality of κ, we conclude that $\dim \Phi_{k+p}(X) = \kappa$. qed

Now, we want to study the asymptotic behavior of $\Phi_{|L^{\otimes m}|}$ for base-point-free line bundles.

If L is a line bundle on X, we defined the set $N(X, L) = \{m \in \mathbb{N} | H^0(X, L^{\otimes m}) \neq 0\}$. In the same way we can define $M(X, L) = \{m \in N(X, L) | L^{\otimes m}$ is globally generated$\}$. Note that, if $L^{\otimes m}$ and $L^{\otimes n}$ are both globally generated, then also their product $L^{\otimes m+n}$ is. Thus, we can define $f = \gcd(m \in M(X, L))$ the **free exponent** of L and, as we did for the exponent, we can see that every large enough multiple of f is in fact in $M(X, L)$.

Remark. If $m \in M(X, L)$, then the linear system $|L^{\otimes m}|$ is base-point-free (see proposition 2.1.5), so the morphism $\Phi_{|L^{\otimes m}|}$ is regular (well-defined everywhere).

Definition 2.4.5. A line bundle L on a projective variety is called **semiample** if $L^{\otimes m}$ is globally generated for some $m > 0$ (i.e. if $M(X, L) \neq \emptyset$).

Let X be a normal projective variety and L a line bundle on it. Recall that we denote $\Phi_m = \Phi_{|L^{\otimes m}|}$ and Y_m the closure of the image of X under Φ_m.

Remark. Let $m > 0$ be an integer such that $L^{\otimes m}$ is globally generated, then $S^k H^0(X, L^{\otimes m})$ determines a base-point-free sublinear system of $|L^{\otimes km}|$ and it corresponds to the Veronese embedding of Y_m. Indeed, if $H^0(X, L^{\otimes m})$ is generated by the set $\{s_0, ..., s_r\}$, $S^k H^0(X, L^{\otimes m})$ is generated by all the monomials in s_i of degree k, which are exactly the monomials defining the k^{th} Veronese embedding of $\mathbb{P}^r$. Therefore, Φ_{km} factorizes through the Veronese embedding of Y_m, i.e. $\Phi_m = \nu_k \circ \Phi_{km}$, with ν_k a finite morphism (since it corresponds to a rational projection between projective spaces as seen in the preliminary results about linear systems and subsystems).

Lemma 2.4.6. *Let X be a normal projective variety and L a semiample line bundle on it. Fix any $m \in M(X,L)$. With the notations above, for sufficiently large k, the composition*

$$X \xrightarrow{\Phi_{km}} Y_{km} \xrightarrow{\nu_k} Y_m$$

gives the Stein factorization of Φ_m (i.e. Φ_{km} is a fibration). In particular, Y_{km} and Φ_{km} are independent from k for k large enough.

Proof. Let $X \xrightarrow{\psi} V \xrightarrow{\mu} Y_m$ be the Stein factorization of Φ_m. Let A_m be the ample line bundle on Y_m which pulls back to $L^{\otimes m}$ on X. The sheaf $B := \mu^* A_m$ is an ample line bundle as μ is finite (see [L1, proposition 1.2.13]). Therefore, for $k \gg 0$, $B^{\otimes k}$ is very ample on V, hence $\Phi_{|B^{\otimes k}|}$ is a birational map with the image. But $\psi^* B^{\otimes k} = L^{\otimes km}$, so that $H^0(X, L^{\otimes km}) = H^0(V, B^{\otimes k})$ by proposition 2.2.9, hence V is birational to the image of Φ_{km}. But this means that $Y_{km} = V$ and $\Phi_{km} = \psi$ for $k \gg 0$. qed

This above is the key lemma to understand the asymptotic behavior of the maps Φ_m for base-point-free line bundles.

Theorem 2.4.7. *Let X be a normal projective variety and L a semiample line bundle on it. Then, there is a fibration $\Phi : X \to Y$ such that, for all large enough $m \in M(X,L)$, $Y_m = Y$ and $\Phi_m = \Phi$. Moreover, there is an ample line bundle A on Y such that $\Phi^* A = L^{\otimes f}$, where f is the free exponent of L (this fact basically means that $L^{\otimes f}$ is trivial on the fibers).*

Proof. Let f be the free exponent of L, by replacing L with $L^{\otimes f}$, we may assume $f = 1$, so every big enough power of L is globally generated. Take m_1 and m_2 two relatively prime integers. By lemma 2.4.6 above, for $k \gg 0$, Y_{km_1} and Y_{km_2} are independent from k. Choose k and k' big enough and such that $p := km_1$ and $q = k'm_2$ are relatively prime, so $\forall h > 0$, $Y_p = Y_{hp}$ and $Y_q = Y_{hq}$. Then, $Y_p = Y_{pq} = Y_q$ and $\Phi_p = \Phi_{pq} = \Phi_q$. Define $\Phi := \Phi_{pq}$ and $Y := Y_{pq}$.
Now we prove that this is the map which gives the statement of the theorem. From the proof of lemma 2.4.6 above, we see that Y carries two ample line bundles A_p, A_q such that $\Phi^* A_p = L^{\otimes p}$ and $\Phi^* A_q = L^{\otimes q}$. But, as p, q are relatively prime, there exist $r, s \in \mathbb{Z}$ such that $1 = pr + qs$. Then, if $A := A_p^{\otimes r} \otimes A_q^{\otimes s}$ (taking a negative power means taking the corresponding positive power of the dual sheaf), $\Phi^* A = L$. Moreover, since Φ^* is injective on the Picard groups (see lemma 2.2.10), $A_p = A^{\otimes p}$ and $A_q = A^{\otimes q}$ and, in particular, A is ample.
For the last step, fix $c, d \geq 1$ two integers. The product $S^c H^0(Y, A_p) \otimes S^d H^0(Y, A_q)$ determines a base-point-free linear subseries of $H^0(Y, A^{\otimes(cp+dq)}) = H^0(X, L^{\otimes(cp+dq)})$. Then, as in the remark preceding the lemma, using the product of two Veronese embeddings, Φ factors through Φ_{cp+dq} and a finite map. But Φ is a fibration, so the finite map must be trivial. Furthermore, any large enough m can be written in the form $cp + dq$, so, $\Phi_m = \Phi_{cp+dq} = \Phi$ and this concludes the proof. qed

Now we are ready to define the Iitaka–Moishezon fibration of a variety for a general line bundle L.

Theorem 2.4.8. *Let X be a normal projective variety and L a line bundle on it such that $\kappa(X,L) > 0$ (the case $\kappa(X,L) = 0$ being trivial). Then, for all $k \in N(X,L)$ large enough, the maps $\Phi_k : X \dashrightarrow Y_k$ are all birationally equivalent to a fixed fibration $\Phi_\infty : X_\infty \to Y_\infty$ of normal varieties and the restriction of L to a very general fiber of Φ_∞ has Iitaka dimension 0. More precisely, for large k, there exists a commutative diagram:*

$$\begin{tikzcd} X \arrow[d, dashed, "\Phi_k"'] & X_\infty \arrow[l, "u_\infty"'] \arrow[d, "\Phi_\infty"] \\ Y_k & Y_\infty \arrow[l, dashed, "\nu_k"] \end{tikzcd}$$

where the horizontal maps are birational and u_∞ is a morphism. We have $\dim Y_\infty = \kappa(X,L)$ and, if we denote by $L_\infty = u_\infty^* L$ and $F \subseteq X_\infty$ a very general fiber of Φ_∞, then $\kappa(F, L_{\infty|F}) = 0$.

Remark. Note that, by proposition 2.2.8, the fibers of a fibration are irreducible, so it makes sense to compute their Kodaira dimension.

Proof. The proof is divided in three steps.

Step 1 The idea of the first step is to reduce the problem to the case in which the sheaf is globally generated in order to use the previous result.
Let $m \in N(X,L)$ an integer such that $\dim Y_m = \kappa(X,L)$. Claim: $\Phi_{km} : X \dashrightarrow Y_{km}$ are all birationally equivalent to a fixed fibration of normal varieties

$$\psi_m : X_{(m)} \to Y_{(m)}$$

for $k \gg 0$.

Let $u_m : X_{(m)} \to X$ be a resolution of the indeterminacy locus of Φ_m with $X_{(m)}$ normal, this map exists by theorem 2.4.3. Let $|u_m^* L^{\otimes m}| = |M_m| + F_m$ be the decomposition of this linear system into its fixed part, F_m and its base-point-free part, $|M_m|$, and let $\psi_m := \Phi_{|M_m|} : X_m \dashrightarrow Y_m$ be the rational map defined by M_m. Note that, by construction, this map factors exactly as $\Phi_m \circ u_m$. Using the base-point-free subseries $|M_m^{\otimes k}|$, define also rational maps $\psi_{km} : X_{(m)} \to Y'_{km}$. Clearly, Y_m is birational to its k^{th} Veronese embedding, and, as we pointed out in the remark preceding the previous theorem, the linear system corresponding to $S^k|M_m|$ gives the Veronese (re-)embedding of Y_m. But it is also a base-point-free subsystem of $|M_m^{\otimes k}|$, so there is a finite map λ_k (see the remark on sublinear systems in the preliminary results) making the diagram commutative:

$$\begin{tikzcd} X_{(m)} \arrow[r, "\Phi_{S^k|M_m|}", "\psi_m"'] \arrow[dr, "\psi_{km}"'] & Y_m \\ & Y'_{km} \arrow[u, "\lambda_k"'] \end{tikzcd}$$

This is exactly the setting we had for the previous theorem 2.4.7, hence we can use that result to say that the maps ψ_{km}, for $k \gg 0$, stabilize to a fixed fibration $\psi_{(m)} : X_{(m)} \to Y_{(m)}$. Furthermore, $|M_m^{\otimes k}|$ is a subsystem of $|u_m^* L^{\otimes km}|$ and u_m is birational, so we can actually consider it as a subsystem of $|L^{\otimes km}|$. Thus, in the same way as before, we get a finite map μ_k, making the following diagram commutative:

$$\begin{tikzcd} X_{(m)} \simeq X \arrow[d, "\psi_{(m)}"'] \arrow[dr, dashed, "\Phi_{km}"] & \\ Y_{(m)} & Y_{km} \arrow[l, dashed, "\mu_k"] \end{tikzcd}$$

But $\psi_{(m)}$ is a fibration, so the finite map μ_k must be trivial, it has to be birational, which means that Φ_{km} is birationally equivalent to $\psi_{(m)}$.

Step 2 The next step is the construction of a common model for the $\psi_{(m)}$. By replacing L with $L^{\otimes e}$, we can assume that the exponent of L, e, is 1. Fix $p, q \gg 0$ relatively prime integers such that $\dim Y_p = \dim Y_q = \kappa(X, L)$ (this can be done as the dimensions stabilize by proposition 2.4.4). Now, choose $m \gg 0$ such that $\psi_{(p)} : X_{(p)} \to Y_{(p)}$ is given by the linear system $|M_p^{\otimes p^{(m-1)}}|$ and $\psi_{(q)} : X_{(q)} \to Y_{(q)}$ by $|M_q^{\otimes q^{(m-1)}}|$. Choose a common model for $X_{(p)}$ and $X_{(q)}$, for example construct X_∞ blowing up X both in the centers used to construct $X_{(p)}$ and in the centers used to construct $X_{(q)}$ and take a resolution of the result.

$$\begin{tikzcd} & X_{(p)} \arrow[dr, "u_p"] & \\ X_\infty \arrow[ur, "v_p"] \arrow[dr, "v_q"'] & & X \\ & X_{(q)} \arrow[ur, "u_q"'] & \end{tikzcd}$$

Let $u_\infty := u_p \circ v_p = u_q \circ v_q$. On X_∞, define the base-point-free line bundle $M_{p,q} := v_p^* M_p^{\otimes p^{(m-1)}} \otimes v_q^* M_q^{\otimes q^{(m-1)}}$. Let Y'_∞ be the closure of $\Phi_{|M_{p,q}|}(X_\infty)$ and Y_∞ its normalization. Call $\Phi_\infty : X_\infty \to Y_\infty$ the corresponding morphism.
Note that $Y_{(p)} = Y'_{p^{(m-1)}p}$ and $Y_{(q)} = Y'_{q^{(m-1)}q}$, where the left hand sides are the closure of the images of the maps ψ_{kp} with $k = p^{(m-1)}$ and ψ_{hq} with $h = q^{(m-1)}$ in the notations of the previous step. All the sheaves involved are globally generated, so the monomials in $M_{p,q}$ correspond to the ones used in the Segre (re-)embedding of $Y_{(p)} \times Y_{(q)}$, thus Y_∞ can be identified with a subvariety of this product and we have the two projection maps onto the two factors. In particular, $\dim Y_\infty \geq \dim Y_{(p)} = \kappa(X, L)$.
Moreover, $|M_{p,q}|$ is a linear subsystem of $|L^{\otimes(p+q)}|$, so there exists a finite map $\varphi : Y_{p+q} \to Y_\infty$ such that $\Phi_{|M_{p,q}|} = \varphi \circ \Phi_{p+q}$. This implies that $\dim Y_\infty = \dim Y_{p+q} \leq$

$\kappa(X, L)$, so $\dim Y_\infty = \kappa(X, L)$.
Consider the commutative diagram (same process also with q instead of p):

$$\begin{tikzcd} X_{(p)} \arrow[d, "\psi_{(p)}"'] & X_\infty \arrow[l, "v_p"'] \arrow[d, "\Phi_\infty"] \\ Y_{(p)} & Y_\infty \arrow[l, "w_p"] \end{tikzcd}$$

The map $\psi_{(p)} \circ v_p = w_p \circ \Phi_\infty$ is surjective, so, in particular w_p is surjective and is a finite map since $\dim Y_{(p)} = \dim Y_\infty$. But $\psi_{(p)} \circ v_p = w_p \circ \Phi_\infty$ is a fibration and the varieties involved are normal, so w_p must be an isomorphism and Φ_∞ is a fibration.
Before the conclusion a comment on this step that will be used later. Let $\epsilon : Y_\infty \hookrightarrow Y_{(p)} \times Y_{(q)}$ and H a hyperplane divisor in $Y_{(p)} \times Y_{(q)}$. Note that, by construction, $A_{p,q} = \epsilon^* H$ is a very ample divisor in Y_∞ such that $\Phi_\infty^* A_{p,q} = M_{p,q}$.
To conclude, fix two integers $c, d \geq 1$. Then, multiplication by a non-zero global section of $u_p^* M_p^{\otimes(c-1)p^{(m-1)}} \otimes u_q^* M_q^{\otimes(d-1)q^{(m-1)}}$ gives an inclusion:

$$H^0(X_\infty, M_{p,q}) \subseteq H^0(X_\infty, u_p^* M_p^{\otimes cp^{(m-1)}} \otimes u_q^* M_q^{\otimes dq^{(m-1)}}) \subseteq H^0(X_\infty, u_\infty^* L^{\otimes(cp^m + dq^m)}),$$

where the last inclusion comes just from the fact that the second system is a subsystem of the third one. So, as usual, we have a finite map μ making the following diagram commutative:

$$\begin{tikzcd} X_\infty \arrow[r, "u_\infty"] \arrow[d, "\Phi_\infty"'] & X \arrow[d, dashed, "\Phi_{cp^m+dq^m}"] \\ Y_\infty & Y_{cp^m+dq^m} \arrow[l, dashed, "\mu"] \end{tikzcd}$$

As Φ_∞ is a fibration, μ has to be birational. Each $k \gg 0$ can be written as $cp^m + dq^m$ since p and q are coprime, thus a rational inverse of μ as above gives the desired diagram in the statement.

Step 3 In this last step we compute the Kodaira dimension of the fibers of Φ_∞.
Let $L_\infty := u_\infty^* L$ and F a fiber of Φ_∞. As X_∞ is birational to X, we can also think of F inside X via u_∞. Note, then, that the image of F under $\Phi_{m|F}$ has obviously non-negative dimension, so $\kappa(F, L_{\infty|F}) \geq 0$. To conclude the proof it is enough to prove the opposite inequality. For $k \gg 0$, we have the diagram:

$$\begin{tikzcd} X_\infty \arrow[r, "\Phi_\infty"] \arrow[d, "u_\infty"'] & Y_\infty \arrow[d, dashed, "\nu_k"] \\ X \arrow[r, dashed, "\Phi_k"'] & Y_k \end{tikzcd}$$

For each $k \gg 0$, let U_k be the complement of the indeterminacy locus of ν_k and $U := \bigcap_{k\gg 0} U_k$. Moreover, let V_k be the complement of the indeterminacy locus of Φ_k and $V := \bigcap_{k\gg 0} V_k$. The fibers for which we can have the conclusion are those $F_y := \Phi_\infty^{-1}(y)$ such that $y \in U$ and $u_\infty(F_y) \subseteq V$. Since we are imposing countably many open conditions, we say that the property holds for a very general point y. Denote $F := F_y$ Observe that $\Phi_k(u_\infty(F)) = \nu_k(y)$ is a point, so

$$\rho_k : H^0(X_\infty, L_\infty^{\otimes k}) \to H^0(F, L_{\infty|F}^{\otimes k})$$

has rank 1 as each global section is constant on F. Our objective is to prove that ρ_k is also surjective. Let B be a very ample line bundle on Y_∞, then $\Phi_\infty^* B$ is ample on X_∞. From the second step, we know there exists $A := A_{p,q}$ ample and globally generated line bundle on Y_∞ such that $\Phi_\infty^* A = M_{p,q}$ on X_∞. Thus, $A^{\otimes m'} \otimes B^{-1}$ has a non-zero section for all $m' \gg 0$. But $M_{p,q}^{\otimes m'}$ is a subsheaf of $L_\infty^{\otimes m'(p^m+q^m)}$, so also $L_\infty^{\otimes m'(p^m+q^m)} \otimes B^{-1}$ has a non-zero global section, call it s. Now fix $k > 0$ and take any $r > 0$. The diagram

$$\begin{tikzcd}
H^0(X_\infty, L_\infty^{\otimes k} \otimes \Phi_\infty^* B^{\otimes r}) \arrow[r, hook, "\cdot s^r"] \arrow[d, "\beta_{k,r}"'] & H^0(X_\infty, L_\infty^{\otimes k+rm'} \otimes \Phi_\infty^* B^{\otimes r} \otimes \Phi_\infty^* B^{\otimes -r}) \arrow[d, "\rho_{k+rm'}"] \\
H^0(F, (L_\infty^{\otimes k} \otimes \Phi_\infty^* B^{\otimes r})_{|F}) \arrow[r, hook, "\cdot s^r_{|F}"] & H^0(F, L_{\infty|F}^{\otimes k+rm'})
\end{tikzcd}$$

is commutative, where $\beta_{k,r}$ is the natural restriction. But:

$$H^0(X_\infty, L_\infty^{\otimes k} \otimes \Phi_\infty^* B^{\otimes r}) = H^0(Y_\infty, \Phi_{\infty,*} L_\infty^{\otimes k} \otimes B^{\otimes r}) \xrightarrow{\beta_{k,r}}$$
$$H^0(F, (L_\infty^{\otimes k} \otimes \Phi_\infty^* B^{\otimes r})_{|F}) = (\Phi_{\infty,*} L_\infty^{\otimes k} \otimes B^{\otimes r}) \otimes \mathbb{C}(y)$$

For $r \gg 0$, as B is ample, $\Phi_{\infty,*} L_\infty^{\otimes k} \otimes B^{\otimes r}$ is globally generated and the restriction map sends each section to its image in the stalk, so it is surjective. As the rank of $\rho_{k+rm'}$ is 1, chasing the diagram, we see that also $\beta_{k,r}$ has rank 1. But, if s is a global section of $B^{\otimes r}$, then $\Phi_{\infty|F}^*(s)$ is constant as $\Phi_\infty(F) = y$, constant. Therefore we finally conclude that $\dim H^0(F, L_{\infty|F}^{\otimes k}) = 1$, which implies that $\kappa(F, L_{\infty|F}) = 0$.

qed

Definition 2.4.9. In the setting of the above theorem, the fibration Φ_∞ is called the **Iitaka–Moishezon fibration** of X, L. It is unique up to birational equivalence.

2.5 The maximal rationally connected quotient

The goal of this section is to understand the construction of the maximal rationally connected (MRC) quotient. This is the "maximal" fibration such that its fibers are rationally connected and its base has non-negative Kodaira dimension, which is conjecturally equivalent to not being uniruled.

Definition 2.5.1. A **rational curve** on a variety X is the image of a regular non-constant map: $\mathbb{P}^1 \to X$.
An n-dimensional variety is called **rational** if it is birationally equivalent to $\mathbb{P}^n$. It is called **unirational** if it is dominated by (i.e. there exists a dominant rational map from) a rational variety.
A variety X is said to be **uniruled** if it is covered by rational curves.
A variety is called **rationally connected** if for any pair of general points there is a rational curve on X containing both.

It turns out that X is uniruled if and only if there exists a dominant rational map $\mathbb{P}^1 \times T \dashrightarrow X$ for a $(n-1)$-dimensional variety T (where n is the dimension of X). Therefore, if X is uniruled, $\kappa(X) \leq \kappa(\mathbb{P}^1 \times T) = -\infty$. The converse is a conjecture in birational geometry, known up to dimension 3.

Conjecture 2.5.2. *Uniruledness conjecture*
A variety X is uniruled if and only if $\kappa(X) = -\infty$.

Proposition 2.5.3. *[K, theorem 3.10.3] Let X be a smooth variety. Then X is rationally connected if and only if it is **chain rationally connected**, i.e. for every pair of general points there exists a chain of rational curves on X containing both.*

Rational varieties are potentially dense, as an open subset is isomorphic to an open subset of the projective space. F. Campana conjectures that rationally connected varieties are potentially dense as well. In the remaining of the section we see how to "separate" the rationally connected part of a variety. This is done with the construction of the MRC quotient. To be able to prove its existence, we need to first introduce some results on constructible sets and a sketch of the construction of the Hilbert space of a variety.

Definition 2.5.4. A **constructible set** is a finite union of locally closed subsets.

Lemma 2.5.5. *[H, ch.II, exercise 3.19] Let $f : X \to Y$ be a morphism of finite presentation of noetherian schemes, then the image of a constructible set is constructible.*

Lemma 2.5.6. *[D, lemma 5.1] Let V be a subset of a noetherian topological space X. If*

$$\overline{V} = V_1 \cup ... \cup V_r$$

is an irredundant decomposition into irreducible components, then:

$$V = (V \cap V_1) \cup ... \cup (V \cap V_r)$$

is also an irredundant decomposition into irreducible components. Furthermore, $\overline{V \cap V_i} = V_i$ for every i.

Lemma 2.5.7. *[D, lemma 5.3] Let $p : X \to Y$ be a flat morphism and W a constructible subset of Y. Then:*

(a) $p^{-1}(\overline{W}) = \overline{p^{-1}(W)}$;

(b) any irreducible component of $p^{-1}(W)$ *dominates an irreducible component of* W*;*

(c) if W *is irreducible and* p *has irreducible fibers, then* $p^{-1}(W)$ *is irreducible.*

Lemma 2.5.8. *[D, 5.5] Let* $p : X \to Y$ *be a morphism whose closed fibers are connected. If* X *is normal and* Y *irreducible, a general fiber of* p *is irreducible.*

Lemma 2.5.9. *[D, 5.4] Let* $p : X \to Y$ *be a morphism with* Y *reduced. Then, there exists an open dense subset* $U \subseteq Y$ *such that* $p : p^{-1}(U) \to U$ *is flat.*

The Hilbert scheme of a variety X is a scheme which parametrizes subschemes of X. For the construction of the MRC quotient we will use a subscheme of the Hilbert scheme parametrizing curves with rational components. Here we sketch the construction of these object, for a more complete presentation see [H-M, ch.I, sections A and B].

Fix X a projective scheme and P a polynomial. Define a functor $\mathrm{H_{P,X}} : \underline{Schemes} \to \underline{Sets}$ which sends a scheme B to the set of proper flat families of subschemes of X, $\mathcal{X} \to B$ where the Hilbert polynomial of $\mathcal{X}$ is P and such they satisfy a commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "i"] \arrow[dr, "\varphi"'] & X \times B \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\ & B & \end{tikzcd}$$

It can be shown that this functor is representable. The projective scheme representing it is called the **Hilbert scheme** of X with polynomial P, denoted by $\mathrm{Hil}_P(X)$. So, for every scheme B, $\mathrm{H_{P,X}}(B) = \mathrm{Hom}(B, \mathrm{Hil}_P(X))$. This means that every family $\varphi : \mathcal{X} \to B$ of subschemes, corresponds uniquely to a morphism $\widetilde{\varphi} : B \to \mathrm{Hil}_P(X)$. We can think at a point $\widetilde{\varphi}(b)$ as the subscheme of X corresponding to the fiber of φ over a point $b \in B$.

If we consider the scheme $B = \mathrm{Hil}_P(X)$, in $\mathrm{Hom}(B, \mathrm{Hil}_P(X))$ there is the identity. The family corresponding to the identity, $\psi : \mathcal{H}_P \to \mathrm{Hil}_P(X)$, is called **universal family** because it satisfies the following universal property: for every family $\varphi : \mathcal{X} \to B$, there is a unique h making the diagram below commutative.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "h"] \arrow[d, "\varphi"'] & \mathcal{H}_P \arrow[d, "\psi"] \\ B \arrow[r, "\widetilde{\varphi}"'] & \mathrm{Hil}_P(X) \end{tikzcd}$$

The union of all these schemes is called **Hilbert scheme** of X and is denoted by $\mathrm{Hil}(X)$.

Later we will need a subscheme of the Hilbert scheme, namely: $\mathrm{Rat}_d(X)$, the projective scheme obtained as the union of all components of $\mathrm{Hil}_P(X)$ whose general points correspond to (reduced connected) curves of degree $\leq d$ with rational components. (The bound on the degree is needed to have a fixed Hilbert polynomial.) In the same way as before we can define the universal family $\mathcal{C}_d$ associated with this subscheme.

The maximal rationally connected quotient is constructed as a quotient by algebraic relations of $\mathrm{Rat}_d(X)$. Let $T, \mathcal{C}$ be reduced quasi-projective schemes and X a projective variety such that there are two maps:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"'] & X \\ T & \end{tikzcd}$$

We define an equivalence relation using these maps. We say that two points $x_1, x_2 \in X$ are equivalent if there exists $\{t_1, ..., t_m\} \subseteq T$ finite such that both x_1 and x_2 belong to a connected component (the same) of $F(\mathcal{C}_{t_1}) \cup ... \cup F(\mathcal{C}_{t_m})$. If this happens we say that x_1 and x_2 are connected by a chain of lenght m. Denote by $[x]_\mathcal{C}$ the equivalence class of x.

Proposition 2.5.10. *In the notations above, suppose that π is flat with irreducible fibers and F is flat. Then, there exists an open dense subset $X' \subseteq X$, a variety Y and a morphism $\tau : X' \to Y$ such that:*

(1) $\tau(x_1) = \tau(x_2)$ if and only if the closures of the two equivalence classes coincide, i.e. $\overline{[x_1]_\mathcal{C}} = \overline{[x_2]_\mathcal{C}}$;

(2) let $x \in X'$, then the general point of $\tau^{-1}(\tau(x))$ can be connected to x by a chain in $\mathcal{C}$ of length $\dim X - \dim Y$.

Proof. Let $x \in X$ and let $V_m(x)$ be the set of points in X that can be joined by a $\mathcal{C}$-chain of length m. Let $\delta(x) = \lim_{m\to\infty} \dim V_m(x)$. As $V_m(x)$ are increasing and bounded by $\dim X$, this limit exists.

Step 1 The goal of this first step is to prove that, if $m \geq \delta(x)$, then $\overline{V_m(x)} = \overline{V_{m+1}(x)}$. First of all we check if $V_m(x)$ is a constructible set. Note that $x, x' \in X$ are joined by a $\mathcal{C}$-chain of length 1 if and only if $\pi(F^{-1}(x)) \cap \pi(F^{-1}(x')) \neq \emptyset$. Indeed, if there exists $t \in T$ such that $x, x' \in F(\mathcal{C}_t)$, then $t \in \pi(F^{-1}(x)) \cap \pi(F^{-1}(x'))$. On the other hand, if $t \in \pi(F^{-1}(x)) \cap \pi(F^{-1}(x'))$, this means that $F^{-1}(x), F^{-1}(x') \cap \mathcal{C}_t \neq \emptyset$, thus $x, x' \in F(\mathcal{C}_t)$. By definition, $V_{m+1}(x)$ is the set of points which can be reached with a $\mathcal{C}$-chain of length 1 from a point in $V_m(x)$. Therefore, $y \in V_{m+1}(x)$ if and only if there exists $x' \in V_m(x)$ such that $\pi F^{-1}(y) \cap \pi F^{-1}(x') \neq \emptyset$, i.e. if and only if $y \in F\pi^{-1}\pi F^{-1}(x')$. So, $V_{m+1}(x) = F\pi^{-1}\pi F^{-1}(V_m(x))$. The claim follows with an induction argument: $V_0(x) = \{x\}$ is constructible and, by lemma 2.5.5, if $V_m(x)$ is constructible, so is $V_{m+1}(x)$.

Now, let $V_m(x) = \bigcup_i V_m^i$ and $F^{-1}(V_m^i) = \bigcup_j W_m^{i,j}$ be irredundant decompositions into irreducible components. The map F is flat and all the sets involved are constructible, so we can apply lemma 2.5.7 to get:

$$F^{-1}(\overline{V_m^i}) = \bigcup_j \overline{W_m^{i,j}}; \quad F(\overline{W_m^{i,j}}) = \overline{V_m^i}.$$

By hypothesis we also have that π is flat with irreducible fibers, thus, as $W_m^{i,j}$ are irreducible, also $\pi(W_m^{i,j})$ are and, using lemma 2.5.7 again, $\pi^{-1}\pi(W_m^{i,j}) := \widetilde{W}_m^{i,j}$ are irreducible as well. By our discussion and using lemma 2.5.6

$$\overline{V_{m+1}(x)} = \bigcup_i F\pi^{-1}\pi\overline{F^{-1}(V_m^i(x))} = \bigcup_{i,j} \overline{F(\widetilde{W}_m^{i,j})}.$$

Note that $\overline{F(\widetilde{W}_m^{i,j})} \subseteq \overline{V_m^i}$. If, for all j, $\overline{F(\widetilde{W}_m^{i,j})} = \overline{V_m^i}$ we say that the component $\overline{V_m^i}$ is **stable**, we call it **unstable** otherwise. Note that $\overline{V_m} = \overline{V_{m+1}}$ if and only if all components $\overline{V_m^i}$ are stable. Let $m \geq \delta(x)$, if $\overline{V_{m+1}(x)}$ has an unstable component, then this is of the form $\overline{F(\widetilde{W}_m^{i,j})}$, so also $\overline{V_m^i}$ must be unstable, otherwise $\overline{F(\widetilde{W}_m^{i,j})} = \overline{V_m^i}$ is stable. Going on in this way, we can construct a chain of (closed) irreducible components which are unstable: $\overline{F(\widetilde{W}_m^{i,j})} \supsetneq \overline{V_m^i} \supsetneq ... \supsetneq V_0(x) = \{x\}$. This implies that the dimension of $\overline{F(\widetilde{W}_m^{i,j})}$ is $\geq m+1 > \delta(x)$. Contradiction. Thus, $\forall m \geq \delta(x)$, $\overline{V_m(x)} = \overline{V_{m+1}(x)}$.

Step 2 In this step we prove that, if $n = \dim X$ (in particular, $n \geq \delta(x)$ for every $x \in X$), then there exists $X' \subseteq X$ open dense such that, given $x, x' \in X'$, $x' \in V_n(x)$ if and only if $\overline{V_n(x)} = \overline{V_n(x')}$.
Let $V = \bigcup_{x \in X} \{x\} \times V_n(x) \subseteq X \times X$. In a similar way as we did in the first step, it can be proven that V is constructible. Let $\overline{V}$ be its closure in $X \times X$ and q be the first projection. By lemma 2.5.9, there exists $X' \subseteq X$ open dense such that $q^{-1}(X') \to X'$ is flat with reduced fibers. Then, by lemma 2.5.7,

$$\overline{q^{-1}(x)} = q^{-1}(x) = \overline{\{x\} \times V_n(x)}.$$

Thus $V_n(x)$ is dense in $q^{-1}(x)$ for all $x \in X'$.
Let $x' \in V_n(x)$, if $y \in V_n(x')$, then $y \in V_{2n}(x)$. By the first step: $\overline{V_n(x')} \subseteq \overline{V_{2n}(x)} = \overline{V_n(x)}$ and, by symmetry of the reasoning, $\overline{V_n(x')} = \overline{V_n(x)}$.
Let p be the second projection from $\overline{V}$; then $\overline{V_n(x)} = \overline{pq^{-1}(V_n(x) \cap X')}$.
Hence, by lemma 2.5.7, $q^{-1}(V_n(x) \cap X')$ is dense in $q^{-1}(\overline{V_n(x)} \cap X')$, so $pq^{-1}(\overline{V_n(x)} \cap X') = \overline{V_n(x)}$.
If $x' \in \overline{V_n(x)} \cap X'$, then $q^{-1}(x') = \{x'\} \times V_n(x')$ and $p(\{x'\} \times V_n(x')) = \overline{V_n(x)}$, whence $\overline{V_n(x)} = \overline{V_n(x')}$.

Step 3 Construction of Y and τ.
Consider the fiber product $\overline{V} \times_X X'$ using $q : \overline{V} \to X$ and call $\varphi : \overline{V} \times_X X' \to X'$. φ induces a map on the Hilbert scheme $\widetilde{\varphi} : X' \to \mathrm{Hil}(X)$ which sends a point x to the point in the Hilbert scheme corresponding to its fiber under φ, $\overline{V_n(x)}$. The variety X' is irreducible and so $\widetilde{\varphi}(X')$ is irreducible as well and constructible by lemma 2.5.5. Let $Y \subseteq \overline{\mathrm{im}(\widetilde{\varphi})}$ dense open. We can factor:

$$\widetilde{\varphi}_{|\widetilde{\varphi}^{-1}(Y)} \to Y \to \mathrm{Hil}(X).$$

Shrink X' to $\widetilde{\varphi}^{-1}(Y)$ and let $\tau := \widetilde{\varphi}_{|\widetilde{\varphi}^{-1}(Y)}$. This is the morphism we are looking for.

Indeed: $\tau(x_1) = \tau(x_2)$ if and only if $\overline{V_n(x_1)} = \overline{V_n(x_2)}$. Since q is flat, its fibers have constant dimension $\dim X - \dim Y$. Moreover, x' belongs to $\tau^{-1}(x)$ if and only if $\overline{V_n(x)} = \overline{V_n(x')}$ and this happens if and only if $x' \in \overline{V_n(x)}$. Thus, the dimension of a general fiber of τ is exactly the dimension of $\overline{V_n(x)}$, which is $\dim X - \dim Y$. Hence, $\delta(x) = \dim X - \dim Y := \delta$ for all $x \in X'$, which means that the general point of $\tau^{-1}\tau(x)$ can be joined to x by a $\mathcal{C}$-chain of length δ.

qed

Actually, we need a stronger result.

Proposition 2.5.11. *[D, theorem 5.9] Let $T, \mathcal{C}$ defining an algebraic relation on X as above, with F, π proper. Then, there exists an open dense subset $X' \subseteq X$ and a morphism $\rho : X' \to Y$ such that each fiber of ρ is a $\mathcal{C}$-equivalence class.*

Now we are ready to construct the MRC quotient.

Theorem 2.5.12. *Let X be a smooth variety. Then X has an MRC quotient, i.e. there exists a rational map*

$$r_X : X \to R(X)$$

such that:

(a) r_X is defined and proper on a dense subset $X' \subseteq X$;

(b) the fibers of $r_{X|X'}$ are rationally connected;

(c) if Z is a normal variety and $\psi : X \dashrightarrow Z$ is a rational map satisfying the first two properties, then there exists a unique $\pi : Z \dashrightarrow R(X)$ such that $r_X = \pi \circ \psi$.

Moreover, very general fibers of the MRC quotient are rationally connected components.

Remark. In the setting of the theorem above, it was recently proven that $R(X)$ is not uniruled ([13, corollary 9.3]).

Proof. Let $\mathcal{C}_m$ be the universal family of $\mathrm{Rat}_m(X)$. Consider the relation given by the natural maps:

$$\begin{array}{ccc} \mathcal{C}_m & \xrightarrow{F_m} & X \\ \Big\downarrow{\scriptstyle \pi_m} & & \\ \mathrm{Rat}_m(X) & & \end{array}$$

The maps F_m and π_m are proper, so we can apply proposition 2.5.11. Therefore, for every m, we get a dense open subset $X'_m \subseteq X$ and a morphism $\rho_m : X'_m \to Z_m$ whose fibers are $\mathcal{C}_m$-equivalence classes, which means they are rationally chain connected by rational curves of degree at most m.
Note that the sequence $\dim Z_m$ is non-increasing as classes get larger with m, so it eventually stabilizes at $\dim Z_{m_0}$. Thus, if $m \geq m_0$, the general fibers of ρ_m have dimension

$\dim X'_m - \dim Z_m = \dim X - \dim Z_{m_0}$. Let Z'_m be a dense open subset where this holds and let $X''_m := \rho_{m_0}^{-1}(Z'_{m_0}) \cap \rho_m^{-1}(Z'_m)$. Note that $x, x' \in X''_m$ are $\mathcal{C}_{m_0}$-equivalent if and only if they are $\mathcal{C}_m$-equivalent. Moreover, as classes increase in m, X''_m is a union of fibers of ρ_{m_0}.
Let $X' := X'_{m_0}$, $X'' := \bigcap_{m \geq m_0} X''_m$, $R(X) := Z_{m_0}$, $r_X := \rho_{m_0}$.
Then, using the fact that smooth varieties are chain rationally connected if and only if they are rationally connected (proposition 2.5.3), we see that $r_{X|X'}$ is regular proper and its fibers are rationally connected. In fact, the fibers of $r_{X|X''}$ (so very general fibers) are rationally connected components because two points in X'' belong to the same rationally connected component if and only if there is a chain of some length of rational curves connecting them, but this is true if and only if they are $\mathcal{C}_{m_0}$-equivalent, i.e. if and only if they belong to the same fiber of r_X.
To conclude we are left to check that the universal property holds.
Let Z be a normal variety and $\psi : X \dashrightarrow Z$ a rational map satisfying properties (a) and (b). Let X_1 be the open dense subset where it is regular.
If $x \in X_1$ is a very general point, then the fibers of ψ are rationally chain connected, so all points in $\psi^{-1}(\psi(x))$ are $\mathcal{C}_m$-equivalent to x for a suitable m. We can take $m \geq m_0$. But this means that $\psi^{-1}(\psi(x)) \subseteq r_X^{-1}(x)$, in other words, r_X contracts the fibers of ψ. But ψ is proper over a neighborhood of x, therefore there exists $Z_0 \subseteq Z$ open neighborhood of $\psi(x)$ such that ρ is well defined on $\psi^{-1}(Z_0)$ and $\psi_* \mathcal{O}_{\psi^{-1}(Z_0)} \simeq \mathcal{O}_{Z_0}$. Then, we can apply lemma 2.2.11 to get the factorization we are looking for. qed

A slight modification of the notion of Kodaira dimension gives an invariant to detect rationally connected varieties.

Definition 2.5.13. A **rational fibration** $f : X \dashrightarrow Y$ is a dominant rational map with irreducible general fibers. Note that, if the map is regular and X and Y are normal, this is equivalent to asking that the fibers are connected.

Definition 2.5.14. Let X be any projective variety. Define:

$$\kappa^+(X) = \max\{\kappa(Y) | \ \exists \text{ a (dominant) rational fibration } f : X \dashrightarrow Y\}.$$

Proposition 2.5.15. *Assuming the uniruledness conjecture 2.5.2, X is rationally connected if and only if $\kappa^+(X) = -\infty$.*
Moreover, the MRC quotient is the unique fibration $g : X \to Z$ such that:

(1) $\kappa^+(X_z) = -\infty$ *for the general fiber* X_z *of* g*;*

(2) $\kappa(Z) \geq 0$.

Proof. If X is rationally connected, it is in particular uniruled. Moreover, if $f : X \dashrightarrow Y$ is dominant, then Y is uniruled as well, so $\kappa(Y) = -\infty$. Conversely, if $\kappa^+(X) = -\infty$, it means that for any rational fibration $f : X \dashrightarrow Y$, $\kappa(Y) = -\infty$. In particular, we can apply this to the MRC quotient. If X is not rationally connected, $R(X)$ has non-negative Kodaira dimension as it is not uniruled (use the uniruledness conjecture 2.5.2), contradiction. To prove the second part of the theorem it is enough to spell out the properties of the MRC quotient using the first part of this theorem. The uniqueness of a fibration will follow from a reinterpretation of the MRC quotient in the next chapter, see proposition 3.1.14. qed

Chapter 3

The core map

The goal of this chapter is the construction of the core map. This fibration conjecturally splits any orbifold pair in its antithetic parts: the base of general type is conjectured to be mordellic, while the special fibers are conjectured to be potentially dense.
In the first section we see some preliminary results from Chow space theory, we will discuss only the ideas, not going into details. In particular, we see what the Chow scheme of a variety is and we see how to construct quotients using it, given a covering family of the variety. Then, we study particular quotients, the $\mathcal{C}$-quotients, where $\mathcal{C}$ is a "stable" class of varieties having a fixed property. We will apply this construction to reinterpret the MRC quotient and generalize it to the orbifold context and to construct both the weak core map and the core map.
In the second section we study the weak core map, which has very similar properties to the core map. But the main problem with this fibration is that it is not preserved by finite étale covers, while we expect potential density and mordellicity to be invariant under them.
For this reason, we need to introduce in our discussion orbifold pairs, they will be used to keep track of multiple fibers. The third section is devoted to discuss the tools we need to be able to use them in our construction.
Finally, in the last section we present the core map and we state some conjectures that use the decomposition it gives. They study the arithmetic properties of the fibers and the base of the core map.

3.1 Decompositions

3.1.1 Some Chow space theory

In this section we present some results on Chow space theory that will be used in the remaining of the chapter. The tools we need are a generalization of what we used for the construction of the MRC quotient. In particular, we discuss without details what is the Chow space associated with a variety, the notions of covering families, Zariski regularity and some decomposition theorems. At the end of the section we see a first application of these results: a reinterpretation of the MRC quotient. This kind of reasoning will be used later also to construct the weak core and the core maps.

Definition 3.1.1. Let X be a complex variety. A d**-cycle** of X is a finite linear combination with integer coefficients of compact irreducible analytic subsets of X of pure dimension d. The **support** of a cycle is the union of all its finitely many components with non-zero coefficient. The set of all d-cycles is denoted by $\mathcal{C}_d(X)$ and the union of all $\mathcal{C}_d(X)$ is denoted by $Chow(X)$ and is called the **Chow scheme** of X.

Thus, any point of the Chow scheme of a variety X parametrizes a cycle. There is an obvious inclusion of the Hilbert scheme inside the Chow scheme which sends each point of the Hilbert scheme, which represents a subvariety of X, to the cycle corresponding simply to that subvariety. By abuse of notation we often refer to a subvariety of X as its corresponding cycle or its point in the Chow scheme.

Remark. The Chow space can be defined also in a way similar to the one we used for the Hilbert scheme. We can define a functor F_X^d which sends a complex space S to the set of "analytic families" of compact d-dimensional cycles parametrized by S. This functor turns out to be representable by a complex space, $\mathcal{C}_d(X)$. Then, the Chow scheme $Chow(X)$ can be defined as the union of these spaces for all $d \geq 0$.

Definition 3.1.2. A subset $S \subseteq Chow(X)$ is called a **covering family** of X if:

1. S is at most a countable union of compact irreducible subvarieties $S_i \subseteq Chow(X)$;
2. for all i, if $s \in S_i$ is a general point, then the cycle associated with s, Z_s, is irreducible and reduced;
3. X is the union of the supports of all the cycles parametrized by S.

Definition 3.1.3. To each subset $S \subseteq Chow(X)$, we can associate its **incidence graph**, which is defined as:

$$G_S := \{(s,x) | s \in S, x \in \mathrm{Supp}(Z_s)\} \subseteq S \times X$$

where Z_s is the cycle associated with s. Note that, if S is a covering family, the second projection from G_S to X is surjective.

Proposition 3.1.4. *[6, theorem 1.7, result originally due to D.Barlet] Let $G \subseteq S \times X$ be an irreducible compact analytic subset such that the restriction of the first projection p on G is surjective. Then, there exists a unique meromorphic map $f : S \dashrightarrow Chow(X)$ sending a general $s \in S$ to the reduced cycle of X with support $p^{-1}(s)$. If, moreover, the fibers of p have all the same dimension and S is normal, then f is holomorphic.*

Definition 3.1.5. Let $S \subseteq Chow(X)$ be a covering family. For $s_1, ..., s_n \in S$, we say that their corresponding cycles $Z_{s_1}, ..., Z_{s_n}$ form an n**-chain** if the union of their support is connected. Two points of X are said to be S**-equivalent** if there exists an n-chain containing both in its support, for some natural number n. This notion defines an equivalence relation on X.

There is a result very similar to the one we already have for the MRC quotient, but for a general covering family S.

Theorem 3.1.6. *[6, theorem 1.1] Let X be a compact connected normal complex space and $S \subseteq Chow(X)$ be a covering family for X. Then, there exists a meromorphic fibration $q_S : X \dashrightarrow X_S$ such that its general fiber is an equivalence class for the relation defined by S. This map is called S-**quotient** of X.*

To be able to state the decomposition results we need, we have to introduce other two technical notions: Zariski regularity and stability of a class of varieties. These are needed because in the following we want to restrict our attention to particular classes of varieties and we consider families with general members in that class.

Definition 3.1.7. Let S be a complex space and $\mathcal{C} \subseteq S$ a subset. Then, $\mathcal{C}$ is said to be **Zariski regular** or **Z-regular** in S if, for every Zariski closed subset $T \subseteq S$, $\mathcal{C} \cap T$ either contains the general points of T, or is contained in a countable union of Zariski closed proper subsets of T.

Remark. Actually this is a very typical situation and counterexamples seem to be unnatural in algebraic or analytic geometry. In particular, this property is always satisfied by the classes we consider (see [5] for a more precise discussion). In the remaining of the chapter we implicitly assume that Z-regularity holds and we omit to mention it.

Lemma 3.1.8. *[6, proposition 2.4] Let $\mathcal{C} \subseteq S$ be Z-regular inside a complex space. Then there exists a countable family of Zariski closed irreducible subsets $S_i \subseteq S$ such that:*

1. *$\mathcal{C}_i := \mathcal{C} \cap S_i$ contains the general point of S_i;*

2. *$\mathcal{C}$ is the union of $\mathcal{C}_i$'s.*

*The sets S_i's are called **components** of $\mathcal{C}$*

Proposition 3.1.9. *Let X be a normal complex space which admits a meromorphic map with meromorphic inverse to a compact Kähler manifold[1] and let $\mathcal{C} \subseteq Chow(X)$ be Z-regular. Denote by T the family of components of $\mathcal{C}$ (if it is not a covering family for X, add the cycles corresponding to the points of X). Let $q_T : X \dashrightarrow X_T$ be the T-quotient of X and let $t \in \mathcal{C}$ be a point such that the corresponding cycle Z_t meets some general fiber of q_T, then Z_t is contained in that fiber. The map q_T is called the $\mathcal{C}$-**reduction** of X.*

Proof. This is immediate from the fact that the fibers of q_T are equivalence classes for the relation induced by T. However, note that T-chains are in general different from $\mathcal{C}$-chains. qed

[1] A Kähler manifold is a manifold with a complex structure, a riemannian structure and also a symplectic structure. All complex projective smooth varieties are examples of compact Kähler manifolds.

Proposition 3.1.10. *[6, theorem 2.7] Let $f : X \to Y$ be an holomorphic fibration between complex spaces with X normal and meromorphic to a compact Kähler manifold. Let $\mathcal{C} \subseteq Chow(X)$ be Z-regular. Let $\mathcal{C}_f \subseteq \mathcal{C}$ be the set of all t's such that the cycle Z_t is contained in some fiber of f. Then:*

- *$\mathcal{C}_f$ is also Z-regular;*
- *f factorizes through the $\mathcal{C}_f$ quotient $q_{\mathcal{C}_f} : X \dashrightarrow X_{\mathcal{C}_f}$. More precisely, there exists $h_{\mathcal{C}_f} : X_{\mathcal{C}_f} \dashrightarrow Y$ such that $f = h_{\mathcal{C}_f} \circ q_{\mathcal{C}_f}$;*
- *for $y \in Y$ general, the restriction of $q_{\mathcal{C}_f}$ to X_y is the $\mathcal{C}_y$-reduction of X_y, where $\mathcal{C}_y := \mathcal{C}_f \cap \mathcal{C}(X_y)$.*

*The map $q_{\mathcal{C}_f}$ is called the $\mathcal{C}$-**reduction** of f.*

3.1.2 The $\mathcal{C}$-quotient

We have now all the ingredients to present the construction of the $\mathcal{C}$-quotient, where $\mathcal{C}$ is a class with a stability property. It decomposes a variety in its part belonging to $\mathcal{C}$, the base, and in its part which has the "opposite" property of the varieties in $\mathcal{C}$. This construction will be used to reinterpret the MRC quotient and to construct both the weak core map and the core map using the classes of not uniruled varieties, general type varieties and general type orbifold pairs respectively.

Let $\mathcal{C}$ be a class of projective varieties[2] with a given property, stable by birational equivalence, and denote by $\mathcal{C}^{\perp}$ the class of all projective varieties which do not admit any rational fibration to a variety $Z \in \mathcal{C}$ with $\dim Z > 0$. The class $\mathcal{C}^{\perp}$ is called the **kernel** of $\mathcal{C}$.

Definition 3.1.11. We say that $\mathcal{C}$ is **stable** if it satisfies the following two properties[3].

(E1) If $f : X \to Z$ is a (regular) fibration with general fiber in $\mathcal{C}$ and $Z \in \mathcal{C}$, then $X \in \mathcal{C}$.

(E2) For a variety X, denote by $\mathcal{C}(X) \subseteq Chow(X)$ the class of subvarieties of X contained in $\mathcal{C}$. For any covering family T of irreducible components of $\mathcal{C}(X)$, the general fiber of the T-quotient, $q_T : X \dashrightarrow X_T$, is in $\mathcal{C}(X)$.

Remark. In the paper [5], the author defines a class to be stable if it satisfies the following ones.

(E1') If $V \subseteq X$ is a subvariety and $f : V \dashrightarrow W$ is a fibration with general fiber V_w in $\mathcal{C}(X)$, and if there exists a subvariety $W' \subseteq V$ such that the point representing the cycle W' is in $\mathcal{C}(X)$ and $f(W') = W$, then the cycle V is in $\mathcal{C}(X)$ as well.

(E2') For any covering family T of irreducible components of $\mathcal{C}(X)$, the general fiber of the T-quotient, $q_T : X \dashrightarrow X_T$, is in $\mathcal{C}(X)$.

[2] The results and conjectures in this chapter can be extended also to compact complex manifolds which are bimeromorphic to some compact Kähler manifold.

[3] E stands for extension.

On the other hand, in the paper [4], the listed properties are more intuitive:

(E1*) If $f : X \to Z$ is a (regular) fibration with general fiber in $\mathcal{C}$ and $Z \in \mathcal{C}$, then $X \in \mathcal{C}$.

(E2*) Let $X \in \mathcal{C}$, and let $T \subseteq Chow(X)$ be a covering family for X, then its general member is in $\mathcal{C}$.

Anyway, for the purpose of our discussion, we thought it is more convenient to keep (E1*) and (E2').

Theorem 3.1.12. *Let $\mathcal{C}$ be a stable class. Then, for any complex projective variety X, there exists a unique fibration $\gamma_X : X \to C_X$ such that:*

1) its general fiber belongs to $\mathcal{C}^{\perp}$;

2) $C_X \in \mathcal{C}$.

If X is defined over a number field k, then so is γ_X.
*γ_X is called the $\mathcal{C}$**-splitting** of X and is functorial. More precisely, any rational fibration $f : X \dashrightarrow Z$ induces a unique rational fibration $\gamma_f : C_X \dashrightarrow C_Z$ such that $\gamma_Z \circ f = \gamma_f \circ \gamma_X$.*

Proof. Sketch. This theorem is proven by induction on $n = \dim X$. If $\dim X = 0$, the statement is immediate as, by convention, $X \in \mathcal{C} \cap \mathcal{C}^{\perp}$.
Let $g : X \to Z$ be a fibration with $Z \in \mathcal{C}$ and $d := \dim Z$ maximal among the varieties Z that have such a fibration. If $\dim Z = 0$, then $X \in \mathcal{C}^{\perp}$ by definition of $\mathcal{C}^{\perp}$. Otherwise, $0 \leq n - d < n$, so we can apply the induction hypothesis on the general fiber. Let $\gamma_z : X_z \to Y_z$ be the $\mathcal{C}$-quotient of the general fiber X_z. By proposition 3.1.10 there exist a fibration $\gamma_{X/Z} : X \to Y$ and a map $h : Y \to Z$ such that $g = h \circ \gamma_{X/Z}$ and the restrictions on the general fibers are exactly the already constructed maps $\gamma_z : X_z \to Y_z$ by their uniqueness. Note that, by construction, $Y_z \in \mathcal{C}$ for the general fiber and $Z \in \mathcal{C}$, so by property (E1), $Y \in \mathcal{C}$. But $\dim Y \geq \dim Z$, so, by maximality of $\dim Z$, $\dim Y = \dim Z$ and the map h is finite. Since g is a fibration, it cannot factorize through a finite map, thus h has to be an isomorphism. Therefore $g = \gamma_{X/Z}$, $Y_z = \{z\}$, $\gamma_z : X_z \to \{z\}$, hence $X_z \in \mathcal{C}^{\perp}$ and the general fibers of g are exactly $X_z \in \mathcal{C}^{\perp}$. Thus, g enjoys the two properties.
Now we prove uniqueness[4]. Let $j : X \to Y$ be a second fibration for which the two properties hold. Let $y \in Y$ general, let $X_y := j^{-1}(y)$ and $Z_y := g(X_y) \subseteq Z$. Consider

$$G := \{(y, z) | y \in Y, z \in Z_y\} \subseteq Y \times Z.$$

Then, by proposition 3.1.4, there exists a covering family of Z parametrized by Y. As $Z \in \mathcal{C}$, by property (E2*), the general $Z_y \in \mathcal{C}$. By the properties of j, $X_y \in \mathcal{C}^{\perp}$, thus it does not admit any fibration towards an element in $\mathcal{C}$. Hence, Z_y must be a point. Therefore, by lemma 2.2.11, there exists a map $h : Y \to Z$, $y \mapsto Z_y$, such that $h \circ j = g$. Thus, $\dim Y \geq \dim Z$. By maximality $\dim Z$, $\dim Y = \dim Z$ and h is a finite map.

[4] For this part, we follow the proof in the paper [4], so we use property (E2*) instead of (E2).

As g is a fibration we conclude that h is an isomorphism.
We are left to prove functoriality. Let $f : X \dashrightarrow Z$ be a rational fibration, let $\gamma_X : X \to C_X$ be the $\mathcal{C}$-quotient of X and $\gamma_Z : Z \to C_Z$ the $\mathcal{C}$-quotient of Z. Let $y \in C_X$, denote by X_y the fiber of γ_X over y and by $C_{Z,y} := \gamma_Z \circ f(X_y)$. Consider the covering family defined using proposition 3.1.4 by the set:

$$G := \{(y, c)|y \in C_X, c \in C_{Z,y}\} \subseteq C_X \times C_{Z,y}.$$

The general fiber $X_y \in \mathcal{C}^{\perp}$. But, by property (E2*), since $C_Z \in \mathcal{C}$, $C_{Z,y} \in \mathcal{C}$. Thus, $C_{Z,y}$ is a point. Therefore, by lemma 2.2.11, we conclude the existence of a map factorizing $\gamma_Z \circ f$ through γ_X.
If X is defined over a number field k, we can do the exact same construction using the class $\mathcal{C}$ restricted to varieties defined over k. Call γ_X^k the quotient constructed with this class. By uniqueness of the $\mathcal{C}$-quotient, the map γ_X must coincide with γ_X^k and so it must be defined over k. qed

As a first application of this construction, we see another (quicker) way to construct the MRC quotient, assuming $C_{n,m}$ and uniruledness conjectures (2.3.3 and 2.5.2 respectively).

Let $\mathcal{C} = \mathcal{K}^{\geq 0}$ be the class of projective varieties with non-negative Kodaira dimension.

Lemma 3.1.13. *The class $\mathcal{K}^{\geq 0}$ is stable.*

Proof. (E1) If $f : X \to Z$, with $\kappa(Z), \kappa(X_z) \geq 0$ for a general fiber, by $C_{n,m}$ conjecture 2.3.3:

$$\kappa(X) \geq \kappa(X_z) + \kappa(Z) \geq 0.$$

(E2) Let $T \subseteq Chow(X)$ be a covering family of X, with $\kappa(X) \geq 0$. Then, consider the T-quotient of X, $q_T : X \dashrightarrow X_T$, by easy additivity theorem 2.3.1, for the general fiber X_t:

$$0 \leq \kappa(X) \leq \kappa(X_t) + \dim X_T,$$

which implies that $\kappa(X_t) \geq 0$, i.e. $X_t \in \mathcal{K}^{\geq 0}$.

qed

Proposition 3.1.14. *For any X, there exists a unique fibration $\rho_X : X \to R(X)$ such that:*

(1) $\kappa^+(X_z) = -\infty$ for the general fiber X_z;

(2) $\kappa(R(X)) \geq 0$.

Proof. Note that $(\mathcal{K}^{\geq 0})^{\perp}$ is the class of varieties X with $\kappa^+(X) = -\infty$ by definition. To conclude, apply theorem 3.1.12 to the class of varieties with non-negative Kodaira dimension. qed

Remark. To conclude that this map we have found is actually the MRC quotient, we need to use the uniruledness conjecture 2.5.2 because a priori we cannot say that all varieties X with $\kappa^+(X) = -\infty$ are rationally connected. Anyway, assuming that conjecture, ρ_X coincides with the MRC quotient. With this construction we have also another property that we did not find before: functoriality.

3.2 The weak core map

In this section we present the weak core map of a variety X. It is a fibration $X \to Z$ with Z of general type and weakly special fibers, conjectured to be mordellic and potentially dense respectively. We will see, however, that this map is not preserved by étale covers and this leads to the definition of the core map.

Definition 3.2.1. A (complex) variety X is called **weakly special** if, for any finite étale cover $u : X' \to X$, there is no rational dominant map $f : X' \dashrightarrow Z$, with Z of general type and $\dim Z > 0$.

Remark. In [11, conjecture 1.2], it is conjectured that a variety is weakly special if and only if it is potentially dense. However, F. Campana conjectures that a slightly different property, namely specialness (see section 3.4 below), is the right characterization of potential density.

Let $\mathcal{C} := \mathcal{K}^{\max}$ be the class of varieties of general type. The weak core map is constructed as the $\mathcal{C}$-quotient using this class.

Lemma 3.2.2. *The class $\mathcal{K}^{\max}$ is stable.*

Proof. (E1) Let $f : X \to Z$ be a fibration with Z of general type and general fibers X_z of general type. Then, by theorem 2.3.2:

$$\kappa(X) = \kappa(X_z) + \dim Z = \dim X_z + \dim Z = \dim X.$$

Thus, X is of general type.

(E2) Let $T \subseteq Chow(X)$ be a covering family of X, with $\kappa(X) = \dim X$. Then, consider the T-quotient of X, $q_T : X \dashrightarrow X_T$, by easy additivity theorem 2.3.1, for the general fiber X_t:

$$\dim X = \kappa(X) \leq \kappa(X_t) + \dim X_T,$$

which implies that $\kappa(X_t) \geq \dim X - \dim X_T = \dim X_t$, i.e. $X_t \in \mathcal{K}^{\max}$.

qed

Definition 3.2.3. The $\mathcal{K}^{\max}$-quotient of a variety X is called the **weak core map** of X.

Now we want to give another description of the weak core, it corresponds to a composition of MRC quotients and Iitaka fibrations, the so called $(J \circ r)^n$**-decomposition**. Let X be a projective variety and let $r : X \to R_X$ be its MRC quotient. Assuming the uniruledness conjecture 2.5.2, $\kappa(R_X) \geq 0$, thus the Iitaka fibration of R_X, $J : R_X \to Y$ is well defined for every X. Iterate this process again and again taking the MRC quotient of Y and then the Iitaka fibration of the quotient, R_Y.

Lemma 3.2.4. *The process described above stabilizes in a finite number of steps, less or equal than* $n := \dim X$. *In particular, the resulting composition* $(J \circ r)^n : X \to X_n$ *is a fibration over a variety* X_n *of general type with fibers that are towers of fibrations with fibers alternately rationally connected or with* 0 *Kodaira dimension.*

Proof. Note that, if $\kappa(Y) \neq -\infty$, then the MRC quotient is the identity. Instead, J is the identity when Y is of general type. Define inductively the i^{th} iterate $(J \circ r)^i : X \to X_i$. Then, $\dim X_{i-1} \geq \dim X_i$, the sequence of dimensions is decreasing and $\dim X_{i-1} = \dim X_i$ if and only if X_{i-1} is of general type. Thus, the process stabilizes when X_i is of general type (from that point on, all the maps are the identity) and this happens in at most $n = \dim X$ steps. The second sentence in the statement of the lemma is then immediate. qed

The claim is that this composition actually coincides with the weak core map. To be able to prove that, we need to study the fibers of the $(J \circ r)^n$-decomposition. We start with a preliminary result.

Proposition 3.2.5. *[5, proposition 2.15] Let* $f : X \dashrightarrow Y$ *and* $h : V \dashrightarrow Z$ *be rational fibrations. Assume that* f *is of general type and assume there is a dominant rational map* $g : V \dashrightarrow X$. *Let* $\gamma_z : V_z \dashrightarrow Y_z := (f \circ g)(V_z)$ *be the restriction of* g *to the general fiber of* h. *If* Y_z *has positive dimension, then the fibration part of the Stein factorization is of general type.*

Theorem 3.2.6. *Let* $f : X \to Y$ *be a fibration with general fiber in* $(\mathcal{K}^{\max})^{\perp}$ *and either* $\kappa(Y) = 0$ *or* $\kappa^+(Y) = -\infty$. *Then* $X \in (\mathcal{K}^{\max})^{\perp}$.

Proof. If X was not in $(\mathcal{K}^{\max})^{\perp}$, then there existed a fibration of general type $g : X \to Z$. Apply proposition 3.2.5 to the diagram:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ Y & Z \end{tikzcd}$$

Consider the restriction of g to the general fiber of f, $g_y : X_y \to Z_y := g(X_y)$. If $\dim Z_y > 0$, the fibration part of the Stein factorization is of general type. But $X_y \in (\mathcal{K}^{\max})^{\perp}$, thus this cannot happen. Therefore Z_y is a point. Applying 2.2.11, we get a map $h : Y \to Z$ such that $h \circ f = g$.

Consider the case when $\kappa(Y) = 0$ and apply theorem 2.3.2 to get, for the general fiber Y_z:

$$0 = \kappa(Y) = \kappa(Y_z) + \dim Z \geq \dim Z > 0.$$

Contradiction. Therefore, $X \in (\mathcal{K}^{\max})^{\perp}$. On the other hand, if $\kappa^+(Y) = -\infty$, by definition such h cannot exist. Thus, also in this case we get a contradiction, whence $X \in (\mathcal{K}^{\max})^{\perp}$. qed

Corollary 3.2.7. *Let $f : X \dashrightarrow Y$ be a fibration with general fiber in $(\mathcal{K}^{\max})^{\perp}$ and X normal variety. Let r be the MRC quotient of Y and J its Iitaka fibration. Then $r \circ f$ and $J \circ f$ have general fibers in $(\mathcal{K}^{\max})^{\perp}$. In particular, by induction, we get that the decomposition $(J \circ r)^n$ of any normal variety has general fiber in $(\mathcal{K}^{\max})^{\perp}$.*

Proof. Firstly, let $r : Y \to Z$ be the MRC quotient of Y. Note that the general fiber of $r \circ f$, X_z, fibers over the general fiber of r, Y_z: $X_z \xrightarrow{f_z} Y_z \to \{z\}$. As $\kappa^+(Y_z) = -\infty$ and the general fiber of f_z belongs to $(\mathcal{K}^{\max})^{\perp}$, theorem 3.2.6 gives the result. Secondly, do the same reasoning with J to conclude. qed

Proposition 3.2.8. *Let $c_X : X \to C_X$ be the weak core map constructed as the $\mathcal{K}^{\max}$-quotient. Assume the uniruledness conjecture 2.5.2, so that the map $(J \circ r)^n$ is well defined. Then $c_X = (J \circ r)^n$. Moreover, we have an explicit description of $(\mathcal{K}^{\max})^{\perp}$: it consists of varieties which are towers over a point of fibrations with fibers either with $\kappa = 0$ or with $\kappa^+ = -\infty$.*

Proof. Both maps have bases in $\mathcal{K}^{\max}$ and general fibers in $(\mathcal{K}^{\max})^{\perp}$ by corollary 3.2.7. Therefore, by uniqueness of the weak core, this two maps must coincide. Now, let $X \in (\mathcal{K}^{\max})^{\perp}$ and let $c_X : X \to C_X$ be its weak core. The variety C_X is a point. As $c_X = (J \circ r)^n$, X is the unique fiber of $(J \circ r)^n$, in particular, it has the described property. qed

Remark. Note that, by the properties of $\mathcal{C}$-quotients, the weak core map is functorial: any fibration $f : X \to Z$ induces a rational fibration $c_f : C_X \dashrightarrow C_Z$. This did not follow immediately from the definition of the $(J \circ r)^n$ map because J is not functorial.

Example 3.2.9.

- Let $H_d \subseteq \mathbb{P}_k^{n+1}$ be an hypersurface of degree d. If $d > n + 2$, it is of general type, so computing the $(J \circ r)^n$-decomposition we see that the weak core map corresponds to the identity. Conversely, if $d = n + 2$, the weak core map sends everything to a point (the Iitaka fibration does so). The remaining case, $d < n+2$, has Kodaira dimension $\kappa(H_d) = -\infty$, but $-K_{H_d}$ is ample, so these hypersurfaces are Fano varieties. By a result of Campana–Kollàr–Miyaoka–Mori, these varieties are rationally connected, so the MRC quotient sends everything to a point and it coincides with the weak core map.

- Let $H_{dd'} \subseteq \mathbb{P}_k^{n+1-j} \times \mathbb{P}_k^j$ be an hypersurface of bidegree d, d'. Let us study the weak core map by computing the $(J \circ r)^n$-decomposition. If $d > n+2-j$ and $d' > j+1$, then it is of general type and so the weak core map is the identity. In the case when $d = n+2-j$ and $d' > j+1$ (or analogously $d > n+2-j$ and $d' = j+1$), the first MRC quotient is the identity and then the Iitaka fibration corresponds to the second projection onto $\mathbb{P}_k^j$, which is rationally connected. Thus, the next MRC quotient sends everything to a point. The resulting weak core map sends everything to a point. When $d = n+2-j$ and $d' = j+1$, the Kodaira dimension is 0, the first MRC quotient is the identity and the Iitaka fibration is onto a point, thus the weak core map is onto a point again. If $d < n+2-j$ and $d' < j+1$, then $-K_{H_{dd'}}$ is ample, so the hypersurface is Fano, thus rationally connected. Thus the MRC quotient is onto a point and it coincides with the weak core map. In the remaining cases ($d < n+2-j$ and $d' \geq j+1$ or viceversa), the Kodaira dimension is $-\infty$, but the computation of the MRC quotient is not so straight-forward.

The weak core map thus decomposes every variety into "pieces" which are either of general type or rationally connected or with 0 Kodaira dimension. General type varieties are conjectured to be mordellic, while rationally connected varieties and varieties with 0 Kodaira dimension are conjectured to be potentially dense. Anyway, there is one problem with this map: it is not preserved by étale covers, while we expect potential density to be preserved by étale covers because of Chevalley-Weil theorem 1.3.2. The example below shows this flaw. F. Campana corrected this by introducing orbifold bases of fibrations to take into account possible multiple fibers and constructing the core map, which can be thought of as an orbifold version of the weak core.

Example 3.2.10. This example shows that the weak core map is not preserved by étale covers, but it also shows an idea of the solution discussed in the next sections: taking into account multiple fibers and therefore working with orbifold pairs.
Let C be an hyperelliptic curve of genus $g \geq 2$. So, we can describe an affine patch of C as $y^2 = f(x)$ for a polynomial $f(x)$ of degree ≥ 3. On C we can consider a particular map, called **hyperelliptic involution**, defined on an affine patch as:

$$\tau : C \to C; \quad (x, y) \mapsto (x, -y).$$

Let $h : C \to \mathbb{P}^1 := C/\langle \tau \rangle$ be the double cover induced by τ, note that it is ramified over the $2g+2$ points images of the hyperelliptic points on C (in the affine patch, they are the point such that $y = 0$). Let E be an elliptic curve, $Q \in E$ a point of order 2 and t the translation by Q, so that t has order 2. Let $S' = E \times C$ and $i = t \times \tau$. Note that i is fixed-point free, so the natural map:

$$u : S' \to S := (E \times C)/i$$

is an unramified double cover.
Since $\kappa(E) = 0$ and $\kappa(C) = 1$, it follows that $\kappa(S') = 1$. The Iitaka fibration of S' must send E to a point and C to a curve birational to C. Thus it coincides with the projection

onto C: $J' : S' \to C$. Since C is of general type, the weak core map coincides with J' and $C'_S = C$.
On the other hand, the Iitaka fibration of S is $J : S \to \mathbb{P}^1 := C/\langle\tau\rangle$. Indeed u is étale, so $\Omega_{S'/S} = 0$ and $\Omega_{S'} = u^*\Omega_S$ by proposition 2.1.14, whence, taking exterior powers, $\omega'_S = u^*\omega_S$. As $H^0(S', \omega'_S) = \langle x \rangle$, x is a generator also of $H^0(S, \omega_S)$. Then, the MRC quotient of $\mathbb{P}^1$ is a point, which is of general type by definition. So, the weak core map sends S to a point and C_S is a point, differently from $C_{S'}$. The induced map $c_u : C_{S'} \to C_S$ does not preserve the dimension.
Let us compute the multiple fibers of J. The map J is ramified with order two over the hyperelliptic points of C. The idea is that, over a general point, the fiber consists of two distinct copies of the curve $E/\langle t \rangle$, one associated with the point $(x,y) \in C$ and one with the point $(x,-y) \in C$. But when $(x,y) = (x,-y)$. i.e. $y = 0$, the fiber consists of only one copy of $E/\langle t \rangle$, which has to be of multiplicity 2. Thus, the "orbifold base" of J is the orbifold pair $(\mathbb{P}^1, \Delta)$, where $\Delta = \sum_{i=1,\dots,2g+2}\left(1 - \frac{1}{2}\right)(p_i)$ with p_i the images of the hyperelliptic points. Note that, by Riemann-Hurwitz' formula, $h^*(K_{\mathbb{P}^1} + \Delta) = K_C$, whence $\kappa(\mathbb{P}^1, K_{\mathbb{P}^1} + \Delta) = \kappa(C, K_C) = 1$, the orbifold base is of general type.

3.3 Orbifold case

In this section we develop the necessary tools to extend the constructions we made to the orbifold case. In particular, we need to define the orbifold base of a fibration, which encodes multiple fibers, the Kodaira dimension of an orbifold pair and what birational equivalence means in the orbifold context. We then reformulate $C_{n,m}$ conjecture in this setting.

Let $f : X \to Y$ be a morphism of varieties over k. Let $y \in Y$ be a point (not necessarily closed) and let I_y be the ideal describing y in an affine patch. Call $J_y = f^*I_y$ the ideal on X describing the fiber over y. The radical of J_y is the intersection of finitely many prime ideals, call them P_i for $i = 1, ..., d$. Let F_i be the irreducible component of the fiber in this affine patch corresponding to P_i. The largest integer m_i such that $J_y \subseteq P_i^{m_i}$ is called the **multiplicity** of F_i.
Let $f : X \dashrightarrow Y$ be a (rational) fibration with X and Y smooth projective varieties. Let $E \subseteq Y$ be an irreducible divisor and let $f^*(E) = \sum_h t_h F_h + R$ be its scheme-theoretic inverse image in X, where in R we collect all components of the fiber whose image in Y has codimension at least 2 and t_h are the multiplicities of the (finitely many) irreducible components F_h. Let $m_f(E) := \inf_h\{t_h\}$, this is called the **multiplicity**[5] of the generic fiber of f over E.

Definition 3.3.1. In the situation above, the orbifold pair (Y, Δ_f), where

$$\Delta_f = \sum_E \left(1 - \frac{1}{m_f(E)}\right) E$$

[5]Classically, this multiplicity is defined with the gcd, but for our purposes it is more appropriate to consider the "inf" multiplicity.

is called the **orbifold base** of f.

Remark. The generic fiber of f is smooth if, affine locally, $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$, where A, B are rings and f induces an extension:

$$B = \frac{A[t_1, ..., t_n]}{(f_1(t_1, ..., t_n), ..., f_m(t_1, ..., t_n))}$$

such that the rank of the jacobian defined by f_i's is maximal. But having a jacobian with maximal rank is an open condition. Thus, if the generic fiber is smooth, the map is smooth in an open dense subset.
In a fibration the generic fiber is smooth, so the set of prime divisors E such that $m_f(E) > 0$ is finite.

As the Iitaka dimension associated with a divisor D deals with the asymptotic behavior of mD for $m \gg 0$, it makes sense to evaluate it also for a $\mathbb{Q}$-divisor. In fact, if D is a $\mathbb{Q}$-divisor, there exists $n \in \mathbb{Z}$ such that nD is a $\mathbb{Z}$-divisor. To compute the Iitaka dimension of D we can consider, then, only sufficiently large multiples of nD.

Definition 3.3.2. Let $f : X \dashrightarrow Y$ be a rational fibration. Define the **Kodaira dimension** of the fibration as:

$$\kappa(Y, f) = \inf_{\bar{f}} \{\kappa(\overline{Y}, K_{\overline{Y}} + \Delta_{\bar{f}})\}$$

where $\bar{f} : \overline{X} \to \overline{Y}$ ranges through all fibrations birationally equivalent to f and $(\overline{Y}, \Delta_{\overline{f}})$ are their orbifold bases.

Remark. If we consider only the "naive" Kodaira dimension of a fibration as the Kodaira dimension of its orbifold base, we do not have birational invariance because we do not have an orbifold version of proposition 2.2.12, which holds only for the usual cotangent bundle.

We want to find an "easier" description of the Kodaira dimension of a fibration. The main problem comes from dealing with divisors whose image has codimension ≥ 2. It turns out that, if we can "get rid" of them (notion of **neat model**), we can compute the Kodaira dimension directly using the orbifold base of the fibration. Another way to compute it, is using the saturation of the sheaf f^*K_Y in Ω_X^p (definition 3.3.5), where $p = \dim Y$.

Definition 3.3.3. Let $f : X \to Y$ be a regular fibration. A prime Weil divisor D on X is said **f-exceptional** if $\mathrm{codim}_Y f(D) \geq 2$. The map f is called **neat** if it is regular, X, Y are smooth and there exists a birational regular map $u : X \to X'$ with X' smooth such that every f-exceptional divisor of X is also u-exceptional.

Proposition 3.3.4. *[5, lemma 1.3] Let $f_0 : X_0 \dashrightarrow Y_0$ be a rational fibration and X' smooth birational to X_0. Then, there exists a neat model $f : X \to Y$ birationally equivalent to f_0 and a birational regular map $u : X \to X'$ such that every f-exceptional divisor is also u-exceptional.*

Definition 3.3.5. Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$, $\mathscr{F}'$ is said to be **saturated** in $\mathscr{F}$ if the cokernel of the inclusion morphism, $\mathscr{F}/\mathscr{F}'$, is torsion free. If it is not saturated, the minimal saturated subsheaf of $\mathscr{F}$ containing $\mathscr{F}'$ is called the **saturation** of $\mathscr{F}'$ in $\mathscr{F}$.
Let $f : X \dashrightarrow Y$ be a rational fibration with X smooth and Y reduced. Define $\mathscr{F}_f$ on X as the saturation of the pull-back of the canonical sheaf on Y, f^*K_Y, in Ω^p_X, where $p = \dim Y$.

Remark. $\mathscr{F}_f$ is a birational invariant, it is preserved by birational modifications of f.

The idea is that we are not working with the "usual" canonical sheaf, but with a "logarithmic version" of it. If $\Delta_f = \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) D_i$ is SNC, then we can assume that D_i are locally the zero locus of the coordinate functions y_i ($r \leq p = \dim Y$). We define locally $\Omega^p_Y(\log \Delta)$ as the invertible sheaf on Y generated by the differential form

$$\frac{dy_1}{y_1^{(1-\frac{1}{m_1})}} \wedge \frac{dy_2}{y_2^{(1-\frac{1}{m_2})}} \wedge \dots \wedge \frac{dy_r}{y_r^{(1-\frac{1}{m_r})}} \wedge dy_{r+1} \wedge \dots \wedge dy_p.$$

Let us consider a simple case to get an idea of what is going on. Let X be a surface and Y a curve. If $F = f^{-1}(P)$ is an irreducible fiber of multiplicity $m > 1$, locally around P we can describe f with the map $(x, y) \mapsto x^m = u$. Thus, $f^*K_Y = f^*du = mx^{m-1}dx$ locally; while $f^*(K_Y + (1 - \frac{1}{m})(P)) = f^*\left(\frac{du}{u^{1-\frac{1}{m}}}\right) = mdx$ which describes indeed the saturation of f^*K_Y.

Proposition 3.3.6. *[5, proposition 1.25] Let $f : X \dashrightarrow Y$ be a rational fibration, with X smooth projective connected variety. Then:*

(a) $\kappa(Y, f) = \kappa(Y, K_Y + \Delta_f)$ *if Y is smooth and f is neat;*

(b) $\kappa(Y, f) = \kappa(X, \mathscr{F}_f)$.

Remark. This result tells us that Δ_f encodes the difference between f^*K_Y and its saturation, in fact, if f is a neat model, by proposition 2.2.9:

$$\kappa(X, \mathscr{F}_f) - \kappa(X, f^*K_Y) = \kappa(Y, K_Y + \Delta_f) - \kappa(Y, K_Y).$$

This is not true for the "classical" notion of orbifold base using the "gcd-multiplicity" and this is the main reason for the introduction of the "inf-multiplicity".

Now we want to go a bit further with the discussion considering the case where the domain is an orbifold pair.

Definition 3.3.7. Let (X, Δ) be an orbifold pair, its Kodaira dimension is defined as the Iitaka dimension of the line bundle $K_X + \Delta$ (or, better, of a suitable multiple of it so that it becomes a $\mathbb{Z}$-divisor). (Recall that in the first chapter we defined the canonical bundle of an orbifold pair to be exactly $K_X + \Delta$.)

Definition 3.3.8. Let (X', Δ') and (X, Δ) be two orbifold divisors. A birational regular map $v : X' \to X$ is said to induce a birational map $v : (X', \Delta') \to (X, \Delta)$ if it is **terminal** with respect to the orbifold structure. More precisely, if $K_{X'} + \Delta' = v^*(K_X + \Delta) + \sum_{j \in J} a_j E_j$, where $a_j \geq 0$ and $\{E_j | j \in J\}$ is the collection of v-exceptional divisors on X'.

Remark. With this definition, the Kodaira dimension of an orbifold pair is a birational invariant.

Definition 3.3.9. An **orbifold étale cover** between two orbifold pairs is a generically finite map between them which is terminal with respect to the orbifold structure (so it can be ramified, but the orbifold divisors "control" the ramification). More precisely, $f : (X, \Delta) \to (Y, H)$ is an orbifold étale cover if it is generically finite and $K_X + \Delta = f^*(K_Y + H) + \sum_{j \in J} a_j E_j$, where $a_j \geq 0$ and $\{E_j | j \in J\}$ is the collection of f-exceptional divisors on X.

Next, we want to define the orbifold base of a fibration when the domain is itself an orbifold pair.
Let (X, Δ) be an orbifold pair, with $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i$. Let $f : (X, \Delta) \to Y$ be a fibration (in the usual sense as a fibration $X \to Y$). For any irreducible divisor $D \subseteq X$, define first its intersection multiplicity with Δ:

$$m(D, \Delta) = \begin{cases} m_i \text{ if there exists } i \text{ such that } D = D_i \\ 1 \text{ elsewhere} \end{cases} .$$

Now, for any $E \subseteq Y$ irreducible divisor, compute:

$$f^*E = \sum_{j \in J} n_j E_j + R$$

where $\{E_j | j \in J\}$ is the collection of irreducible components of f^*E mapped surjectively to E by f, n_j is their scheme-theoretic multiplicity in the pull-back and R is f-exceptional. Define finally the multiplicity of f along E as:

$$m_{f,\Delta}(E) := \inf_{j \in J} \{n_j m(E_j, \Delta)\}$$

Definition 3.3.10. Let $f : X \to Y$ be a fibration with Y smooth and let Δ be an orbifold structure on X. The **orbifold base** of the induced fibration $f : (X, \Delta) \to Y$ is the divisor

$$\Delta_{f,\Delta} := \sum_{E \subseteq Y} \left(1 - \frac{1}{m_{f,\Delta}(E)}\right) E.$$

Remark. Let $f : X \to Y$ and $g : Y \to Z$ be two fibrations with X, Y, Z smooth. Let $H = \Delta_f$, then it is not true that $\Delta_{g,H} = \Delta_{g \circ f, \Delta}$. We can only say that

$$\Delta_{g,H} \geq \Delta_{g \circ f, \Delta},$$

where we say that a divisor A is bigger than a divisor B, $A \geq B$, if the difference $A - B$ is an effective divisor.
Indeed, let us compute the multiplicity of the irreducible divisors appearing in both sides. Let $D \subseteq Z$ be a prime divisor, then:

$$m_{g,H}(D) = \inf_{i \in I}\{m_i m(H, D_i)\}$$

where $g^*D = \sum_{i \in I} m_i D_i + R$. Whereas:

$$m_{g \circ f, \Delta}(D) = \inf_{k \in K}\{n_k\}$$

where $(g \circ f)^*(D) = \sum_{k \in K} D_k + R'$. But $f^*D_i = \sum_{k \in K_i} \widetilde{m}_{i,k} D_k + R_i$, so $n_k = \widetilde{m}_{i,k} m_i$ and $\inf_{k \in K_i} n_k = m_i m(H, D_i)$. To conclude, note that $\bigcup_{i \in I} K_i \subseteq K$, but K may contain another subset J, parametrizing irreducible components of $(g \circ f)^*(D)$ which are mapped surjectively to D, but are f-exceptional. Thus:

$$m_{g \circ f, \Delta}(D) = \inf\{\inf_{k \in \bigcup_i \in I K_i}\{n_k\}, \inf_{k \in J}\{n_k\}\} = \inf\{\inf_{i \in I}\{m_i m(H, D_i)\}, \inf_{k \in J}\{n_k\}\} \leq m_{g,H}(D).$$

Definition 3.3.11. As before, we can define the Kodaira dimension of a fibration $f : (X, \Delta) \to Y$ as

$$\kappa((X, \Delta), f) := \inf_{\bar{f}}\{\kappa(\overline{Y}, K_Y + \Delta_{\bar{f}, \overline{\Delta}})\}$$

where $\bar{f} : (\overline{X}, \overline{\Delta}) \to \overline{Y}$ ranges over all fibrations birationally equivalent to f with domains orbifold birationally equivalent.

There is an analogous way to compute directly the Kodaira dimension of a fibration using the saturation, but this time inside the "log differentials". In fact, $\kappa((X, \Delta), f)$ can be computed as the Iitaka dimension on X of the line bundle $\mathscr{F}_{f,\Delta}$ defined as the saturation of f^*K_Y in $\Omega^p_X(\log \Delta)$.

Finally, we restate "easy additivity" results/ conjectures in an orbifold version.

Definition 3.3.12. Let $f : X \to Y$ be a fibration with X and Y smooth, f is called **prepared** if the lucus $Y' \subseteq Y$ of points y with smooth fiber X_y has a complement contained in a normal crossing divisor D such that $f^{-1}(D)$ is also a normal crossing divisor. (It is possible to show that any fibration has an equivalent prepared model.)

Definition 3.3.13. Let $f : (X, \Delta) \to Y$ be a morphism, f is called **high** if there exists a birational regular $u_0 : X \to X_0$ with X_0 smooth, such that $\kappa(X, K_X + \Delta) = \kappa(X_0, K_{X_0} + (u_0)_*(\Delta))$ and such that every g-exceptional divisor is also u_0-exceptional.

Conjecture 3.3.14. $C^{orb}_{n,m}$
Let $f : (X, \Delta) \to Y$ be a (regular) fibration between smooth varieties. Assume that f is prepared and high, then, for the general fiber $(X_y, \Delta_y := \Delta_{|X_y})$:

$$\kappa(X, K_X + \Delta) \geq \kappa(X_y, K_{X_y} + \Delta_y) + \kappa(Y, K_Y + \Delta_{f,\Delta}).$$

Theorem 3.3.15. *[4, theorem 7.10, result due to E. Viehweg adapted in this context by F. Campana] Let $f : (X, \Delta) \to Y$ be a (regular) fibration between smooth varieties. Assume that f is prepared and high and of general type, i.e. $\kappa(Y, K_Y + \Delta_{f,\Delta}) = \dim Y$. Then, for a general fiber X_y,*

$$\kappa(X, K_X + \Delta) = \kappa(X_y, K_{X_y} + \Delta_y) + \dim Y$$

3.4 The core map

In this section we see the construction of the core map, which is quite similar to the construction of the weak core, but it takes into account multiple fibers with the help of the introduction of orbifolds in the discussion. This map is invariant by finite étale orbifold covers and conjecturally divides the potentially dense and the mordellic part of a variety.

It is conjectured that the potentially dense variety are exactly the **special** varieties, so we start by discussing this notion.

Definition 3.4.1. A variety X is called **special** if $\kappa(X, L) < p$, for any line bundle $L \subseteq \Omega^p_X$ (the p^{th} alternating powers of the sheaf of differentials) and any $p > 0$.

Example 3.4.2. Note that $\kappa(X, L) \leq n := \dim X$, so it is enough to check the property for $p \leq n$.

(1) If X is a curve, the only possible line bundle is $L = K_X$. Thus, a curve is special if and only if it is not of general type. In other words, it is special if it is either rational or elliptic.

(2) If X is rationally connected, it can be proven that $h^0(X, \Omega_X^{\otimes m}) = 0$, for all $m > 0$. Therefore X is special.

(3) If X is of general type, it is not special. In fact, taking $L = \Omega^n_X = K_X$, gives $\kappa(X, L) = n$.

(4) If there exists a rational fibration $f : X \dashrightarrow Y$, where Y is of general type of dimension p, then X is not special. Indeed, by proposition 2.1.14, $f^*\Omega_Y \subseteq \Omega_X$ and these sheaves are locally free. Thus, taking exterior powers is exact and we get $L := f^*K_Y \subseteq \Omega^p_X$. Then, $\kappa(X, L) = \kappa(Y, K_Y) = p$ by proposition 2.2.9.

(5) Let Y be a variety of general type with $\dim Y = k$ and Z a variety of positive dimension, with $\kappa(Z) = 0$. Consider $X := Y \times Z$, then, $L := K_Y \subseteq \Omega^k_X$, as $\Omega_X = \Omega_Y \oplus \Omega_Z$ and $\kappa(X, L) = k$, which shows that X is not special, but has Kodaira dimension $\kappa(X) = k \in \{1, ..., n-1\}$. There can be found also examples of special varieties X with $\dim X = n$ and $\kappa(X) = k \in \{1, ..., n-1\}$. We can therefore conclude that the Kodaira dimension does not characterize specialness directly.

Theorem 3.4.3. *[4, theorem 7.4]*, result originally from *[3] Let $L \subseteq \Omega_X^p$ be a line bundle. Then:*

(i) $\kappa(X, L) \leq p$*;*

(ii) If $\kappa(X, L) = p$, there exists a fibration $f : X \dashrightarrow Z$ such that $f^*(K_Z) = L$ on a non-empty Zariski open subset of X.

Corollary 3.4.4. *If $\kappa(X) = 0$, then X is special.*

Proof. If it was not true, by the previous theorem 3.4.3, there existed a fibration $f : X \dashrightarrow Z$ such that $f^*(K_Z) = L \subseteq \Omega_X^p$ and $\kappa(X, L) = p = \dim Z$. Thus, Z is of general type and we can apply theorem 2.3.2 to get, for a general fiber X_z:

$$0 = \kappa(X) = \kappa(X_z) + \dim Z \geq \dim Z > 0.$$

Contradiction. (We must have $\kappa(X_z) \geq 0$, otherwise also $\kappa(X)$ should be $-\infty$ by the same formula.) qed

Corollary 3.4.5. *A variety X is special if and only if, for any fibration $f : X \dashrightarrow Z$, the orbifold base of any of its neat models is not of general type.*

Proof. If X is special, then, for any neat model of the fibration, by proposition 3.3.6 $\kappa(Z, K_Z + \Delta_f) = \kappa(X, L) < p$, where L is the saturation of f^*K_Z in Ω_X^p ($p = \dim Z$). Thus, the orbifold base is not of general type.
Conversely, if f was not special, there existed $L \subseteq \Omega_X^p$ such that $\kappa(X, L) \geq p$. By theorem 3.4.3, $\kappa(X, L) = p$ and there existed a fibration $f : X \dashrightarrow Z$ such that $L = f^*K_Z$ generally. Thus, the orbifold base of f is of general type. Contradiction. qed

Finally, we have all the ingredients we need to construct the core map. It is constructed using an orbifold modification of theorem 3.1.12. The stable class $\mathcal{C}$ now is a class of smooth orbifold pairs and its kernel is defined as the class $\mathcal{C}^\perp$ of smooth orbifold pairs admitting no rational fibration such that a neat model of it has orbifold base in $\mathcal{C}$. Theorem 3.1.12 holds also in this context with the same proof.

Lemma 3.4.6. *The class $\mathcal{C} := \mathcal{K}_{orb}^{\max}$ of orbifold pairs of general type is stable.*

Proof. This proof is done exactly like the one for lemma 3.2.2, using the orbifold version of easy additivity when the base is of general type 3.3.15 and easy additivity theorem 2.3.1 for the orbifold canonical bundle. qed

Definition 3.4.7. Applying the orbifold version of theorem 3.1.12 to the class $\mathcal{C}$, we obtain a unique fibration $c_X : X \to C_X$ called the **core** of X, such that:

- its general fibers are special (consequence of corollary 3.4.5);
- its orbifold base (C_X, Δ_{C_X}) is of general type;

- any dominant map $g : Y \dashrightarrow X$ induces a map on the cores $c_g : C_Y \to C_X$ such that $c_X \circ g = c_g \circ c_Y$;
- if X is defined over a number field k, so is its core by uniqueness.

We prove now that the core map is preserved by finite étale covers. However, before that, we need a technical result.

Proposition 3.4.8. *[5, theorem 1.8] Let $f : X \to Y$ and $f' : X' \to Y'$ be two fibrations and let $u : X' \to X$ and $v : Y' \to Y$ be two maps such that $f \circ u = v \circ f'$. Assume that u and v are generally finite and surjective, then $\kappa(Y', f') \geq \kappa(Y, f)$. If, moreover, u is étale and X, X' are smooth, then $\kappa(Y', f') = \kappa(Y, f)$.*

Proposition 3.4.9. *If $u : X' \to X$ is finite étale, $c_u : C_{X'} \to C_X$ is generally finite (ramified, but orbifold étale). In particular, if X is special, so is X'.*

Proof. Consider the Galois closure of u, namely $u' : X'' \to X' \to X$. If we can prove that the statement is true for u', then it must be true also for u because, by uniqueness, the map $c_{u'}$ factors through c_u. We can therefore assume that u is Galois. Let $G = \mathrm{deck}(u)$. The group G acts on X', so each element $g \in G$ induces a (dominant) map $g : X' \to X'$, which induces a map at the level of the core: $c_g : C_{X'} \to C_{X'}$. As $u \circ g = u$, $c_X \circ u \circ g = c_X \circ u = c_u \circ c_{X'} = c_u \circ c_g \circ c_{X'}$ by commutativity of the diagram with the core maps.
Let $h : C_{X'} \to C_{X'}/G$ be the quotient map induced by this action. Given $x \in X$, $u^{-1}(x) = \{gx_0 | g \in G\}$ for a fixed x_0 since u is Galois, so G acts transitively. By commutativity, $c_{X'}(gx_0) = c_g c_{X'}(x_0)$, thus $h(c_{X'} gx_0)$ is constant for all $g \in G$. We can then define a map $c'_X : X \to C_{X'}/G$. The fibers of this map are special. Indeed, if $y \in C_{X'}/G$, $h^{-1}(y) = \{gy | g \in G\}$ and $Y_g = c_{X'}^{-1}(gy)$ are special. The map u sends "locally isomorphically" each of these Y_g to the same subvariety of X, which is then special and is the fiber over y of c'_X.
During the proof of theorem 3.1.12, we saw that, when we are in this situation, we can construct $v : C_{X'}/G \dashrightarrow C_X$ and $c_X = v \circ c'_X$. By proposition 3.4.8, as $c_{X'}$ is of general type, also c'_X is so and it has special fibers. Therefore, applying proposition 3.2.5 in the same way we used it in theorem 3.2.6, there exists $w : C_X \to C_{X'}/G$. Thus $c_X = v \circ w \circ c_X$ and $c'_X = w \circ v \circ c'_X$, but c_X and c'_X are dominant, hence v and w are inverse to each other. We can therefore conclude that, birationally, C_X coincides with $C_{X'}/G$ and $h = c_u$ is finite as the group G is so.
Now, let X be special, then C_X has dimension 0 and, as c_u is generically finite, $C_{X'}$ has also dimension 0. Therefore X' is special as well. qed

A consequence of this proposition is that special varieties are, in particular, weakly special.

Corollary 3.4.10. *If X is special, it is weakly special.*

Proof. Let X be special and assume, by contradiction, that it was not weakly special. Then, it existed a finite étale cover $u' : X' \to X$ and a dominant rational map $f' : X' \dashrightarrow$

Y' with Y' positive dimensional of general type.
As in the previous proof, we can reduce to the Galois case. Indeed, consider $u : X'' \to X' \to X$ the Galois closure of u'. Composing $X'' \to X'$ and f' we obtain a dominant rational map of X'' towards a positive dimensional variety of general type, call this map f' again by abuse of notation. We can, therefore, work with u, which is Galois, instead of u'.
Let $G := \mathrm{deck}(u)$ be the group of deck transformations. First, assume f' is G-equivariant (i.e. if there exist $x_1, x_2 \in X''$ such that $f'(x_1) = f'(x_2)$, then $f'(gx_1) = f'(gx_2)$ for all $g \in G$). Then, the action of G on X'' induces an action on Y'. There are natural maps $v : Y' \to Y'/G$ and $f : X \to Y'/G$, towards the quotient by this action such that $f \circ u = v \circ f'$. As u is étale and v is generally finite, by proposition 3.4.8, we can conclude that f' is of general type as well. Contradiction.
On the other hand, if f' is not G-equivariant, we substitute it with f'', a G-equivariant map that we will construct below and we apply the same reasoning to that.

Enumerate the elements in $G = \{g_1, ..., g_N\}$ and consider the finite family $(f_i := f' \circ g_i)_{g_i \in G}$ with the ordering $f_i : X'' \to Y_i \geq f_j : X'' \to Y_j$ if there exists a meromorphic fibration $h : Y_i \dashrightarrow Y_j$ such that $h \circ f_i = f_j$. The map f'' will be the least upper bound of this family with respect to this order. It can be constructed as the fibration part of the Stein factorization of the map:

$$f_1 \times f_2 \times ... \times f_N : X'' \to Y' \times ... \times Y'.$$

Let $Y'' := f''(X')$. We can show that f'' is of general type. Proceeding by induction, it is enough to prove the claim for $N = 2$. Note that the two projections from Y'' to Y' are both surjective and finite and their bases are of general type. Thus we can apply theorem 2.3.2 to one of the projections to get, for the general fiber Y''_y:

$$\kappa(Y'', f'') = \kappa(Y''_y) + \dim Y' = \dim Y''.$$

Whence f'' is of general type. We claim that f'' is G-equivariant. Indeed, if $f''(x_1) = f''(x_2)$, then, by construction, $g_i x_1 = g_i x_2$ for all i. Therefore, $(f_1 \times f_2 \times ... \times f_N)(gx_1) = (f_1 \times f_2 \times ... \times f_N)(gx_2)$, which gives the result. qed

Next, we want to interpret the core map as an orbifold version of the $(J \circ r)^n$-decomposition.
Let (X, Δ) be an orbifold pair, define j as the Iitaka fibration associated with (a big enough power of) the $\mathbb{Q}$-line bundle $K_X + \Delta$ if $\kappa(X, K_X + \Delta) \geq 0$. Thus, $j : (X, \Delta) \to (Y, \Delta_{j,\Delta})$ is a fibration with $\dim Y = \kappa(X, K_X + \Delta)$ and $\kappa(X_y, K_{X_y} + \Delta_{|X_y}) = 0$ for the general fiber X_y of j.
Then, we need an analogue of the MRC quotient and we construct it using a $\mathcal{C}$-decomposition in the orbifold version.

Definition 3.4.11. Let (X, Δ) be an orbifold pair. Define

$$\kappa^+(X, K_X + \Delta) := \max_f \{\kappa(Z, \Delta_{f,\Delta})\}$$

where $f : (X, \Delta) \dashrightarrow Z$ is a neat model of a fibration from the orbifold pair.

F. Campana conjectures an orbifold analogue of uniruledness conjecture, to understand it, we need to suitably modify the notions of rational curve and rational connectedness.

Definition 3.4.12. A **curve** in an orbifold pair $(X, \Delta = \sum \left(1 - \frac{1}{m_j}\right) D_j)$ is a regular orbifold morphism $h : C \to (X, \Delta)$ from a curve C. More precisely, it is a regular morphism such that:

(i) $h(C)$ is not contained in the support of the orbifold divisor Δ;

(ii) for any $a \in C$ and any j such that $h(a) \in D_j$, so $h^*(D_j) = t_{a,j}(a) + ...$, then we ask that $t_{a,j} \geq m_j$ (or, for the "classical version", we ask divisibility by m_j).

Definition 3.4.13. Let (X, Δ) be a smooth orbifold pair, with X complex projective. Then (X, Δ) is **κ-rationally connected** if any two general points of X are contained in an orbifold rational curve $h : \mathbb{P}^1 \to (X, \Delta)$.

Conjecture 3.4.14. *Let (X, Δ) be a smooth orbifold pair with X complex projective. Then (X, Δ) is κ-rationally connected if and only if $\kappa^+(X, K_X + \Delta) = -\infty$.*

Remark. This conjecture is an orbifold version of proposition 2.5.15, which relies on the uniruledness conjecture.

To construct the orbifold version of the MRC quotient, consider the class $\mathcal{C} = \mathcal{K}_{\text{orb}}^{\geq 0}$ of orbifold pairs with positive orbifold Kodaira dimension.

Lemma 3.4.15. *The class $\mathcal{K}_{orb}^{\geq 0}$ is stable.*

Proof. This proof goes in the exact same way as the proof of lemma 3.1.13, using the orbifold version of $C_{n,m}$ conjecture, 3.3.14, and theorem easy additivity 2.3.1 for the orbifold canonical bundle. qed

Proposition 3.4.16. *Any smooth orbifold pair (X, Δ) admits a unique fibration $r : (X, \Delta) \to (R, \Delta_{r,\Delta})$ such that:*

(i) $\kappa^+(X_r, K_{X_r} + \Delta_{|X_r}) = -\infty$ for the general fiber X_r (thus, the fibers are conjectured to be rationally connected);

(ii) $\kappa(R, K_R + \Delta_{r,\Delta}) \geq 0$.

*The map r is called the **κ-rational quotient** of (X, Δ) and is the analogous of the previous MRC quotient.*

Proof. Note that $(\mathcal{K}_{\text{orb}}^{\geq 0})^{\perp}$ coincides with the class of orbifold pairs with $\kappa^+ = -\infty$ by definition. To get the result, apply the orbifold version of theorem 3.1.12 to the class $\mathcal{K}_{\text{orb}}^{\geq 0}$. qed

We now want to describe the core map as a composition of subsequent κ-rational quotients and Iitaka fibrations, as we already did for the weak core map.

Let X be a normal variety, consider its κ-rational quotient $r : X \to R_X$. Then, using orbifold uniruledness conjecture 3.4.14, we get that the orbifold Kodaira dimension of R_X is non-negative. The Iitaka fibration for the orbifold canonical bundle is, thus, well-defined. Repeat this process until the maps stabilize to the identity. With the same reasoning as in the non-orbifold situation, it can be seen that this process stops when the resulting base is of general type, after at most $n := \dim X$ steps. The resulting map is called the $(j \circ r)^n$**-decomposition**.

With the next results, we will prove that the fibers of this decomposition are special, which, by corollary 3.4.5, are exactly the elements in the class $(\mathcal{K}^{\max}_{\text{orb}})^{\perp}$.

Theorem 3.4.17. *Let $f : X \to Y$ be a fibration with general special fiber. Assume that either $\kappa(Y, K_Y + \Delta_f) = 0$ or $\kappa^+(Y, K_Y + \Delta_f) = -\infty$, then X is special.*

Proof. This proof is done in the exact same way as the one for theorem 3.2.6, considering suitable "good models" of the involved fibrations to compute their Kodaira dimension and using the orbifold version of easy additivity with base of general type 3.3.15. qed

Corollary 3.4.18. *Let $f : X \dashrightarrow Y$ be a fibration with special general fiber and X a normal variety. Let r be the MRC quotient of Y and j its orbifold Iitaka fibration. Then $r \circ f$ and $j \circ f$ have special general fibers. In particular, by induction, we get that the decomposition $(j \circ r)^n$ of any normal variety has special general fiber.*

Proof. The proof goes exactly as its twin in the non-orbifold case, using "good models" for the involved fibrations. qed

Theorem 3.4.19. *Let $c_X : X \to C_X$ be the core map of a smooth connected projective variety of dimension n. Then, assuming $C^{orb}_{n,m}$ conjecture 3.3.14 and the orbifold uniruledness conjecture 3.4.14, $c_X = (j \circ r)^n$, where j is the Iitaka fibration relative to the orbifold canonical bundle and r is the κ-rational quotient defined above. In particular, a variety X is special if and only if it is a tower of fibrations over a point with orbifold fibers having either $\kappa^+ = -\infty$ or $\kappa = 0$.*

Proof. The composition $(j \circ r)^n$ has special fibers by corollary 3.4.18, therefore it must coincide with the core map by uniqueness of a map with base of orbifold general type and fibers in the kernel of the class $\mathcal{K}^{\max}_{\text{orb}}$. Besides, X is special if and only if its core map is onto a point, thus the decomposition of the core as $(j \circ r)^n$ gives the result. qed

Now, we come to the main conjecture formulated by F. Campana. It says that the core map splits any orbifold pair in its mordellic part (the base of the core) and in its potentially dense part (the fibers). We formulate it for varieties, but its possible solution should consider orbifold pairs.

Conjecture 3.4.20. *Let X be a variety defined over a number field k and let $c_X : X \to C_X$ be its core map. Then, there exists a complex projective subvariety $W \subsetneq C_X$*

such that, for any finite extension k'/k, $c_X(X(k')) \cap U$ is finite, where $U := C_X \setminus W$. Moreover, there exists k' such that, for any $k'' \supseteq k'$, $X(k'')$ is Zariski dense in each fiber of c_X lying over $c_X(X(k'')) \cap U$.

Conjecture 3.4.21. *Let (X, Δ) be a smooth projective orbifold pair over a number field k.*

- *If (X, Δ) is of general type, then there exists a Zariski closed subset $W \subsetneq X$ such that, for any pair k', S', where k'/k is a finite extension and S' is a finite set of non-archimedean places of k', the set of (S', Δ)-integral points, $(X, \Delta)(S', k')$, contained in $X \setminus W$ is finite.*
- *If $\kappa(X, \Delta) = 0$ or $\kappa^+(X, \Delta) = -\infty$, then there exists k', S', finite extension of k and finite set of non-archimedean places of k' respectively, such that the (S', Δ)-integral points are Zariski dense in X.*

Remark. The first part of the above conjecture is a generalization of Lang's and Vojta's conjectures in the orbifold context.

Remark. Consider two more properties: the orbifold birational invariance of potential density and mordellicity and the fact that, if the fibers and the base of a fibration are potentially dense, so is the domain. These two together with the second part of the conjecture above 3.4.21 and the characterization of specialness in corollary 3.4.19, would imply that the class of special orbifold pairs consists exactly of the potentially dense ones.

Example 3.4.22. We see now an example that shows that we cannot remove the hypothesis of simple normal crossing on the divisors we are considering.
Consider the projective plane from which we remove the divisor D defined by the equations $XY + Z^2 = YZ$ and $ZX = 0$, it is not of simple normal crossing as, around the point $[0 : 1 : 0]$ we need at least $3 > \dim \mathbb{P}^2$ linear parameters to define the divisor. In the usual affine $Z \neq 0$, the integral points on the complement of the two lines correspond to pairs (x, y) with x an S-unit and $y \in \mathcal{O}_{k,S}$. The condition that the conic imposes is that $v := xy - y + 1$ must be an S-unit. This can be rewritten as:

$$\frac{v-1}{x-1} \in \mathcal{O}_{k,S}$$

with x, v S-units. The solutions of this problem are Zariski dense in the plane. For instance, there exists infinitely many pairs (m, n) such that

$$\frac{3^m - 1}{2^n - 1} \in \mathbb{Z}.$$

It is, in fact, sufficient to take an odd number n, so that $2^n - 1$ and 3 are coprime (2^n is congruent to -1 modulo 3 for odd n) and set m to be the order of 3 modulo $2^n - 1$. However, the pair $(\mathbb{P}^2, D)$ is of general type as the degree of the divisor $K_{\mathbb{P}^2} + D$ is $1 > 0$, thus $K_{\mathbb{P}^2} + D$ is very ample. The conjectures we saw, then, would imply that this orbifold pair is mordellic.

We end this chapter with another conjectural interpretation of special varieties: they generalize the notion of rationally connected varieties, substituting rational curves with entire curves.

Definition 3.4.23. An **entire curve** in a complex projective variety X is a non-constant holomorphic map $h : \mathbb{C} \to X$. Note that this is a generalization of the notion of rational curves since here we consider holomorphic maps, among which there are the algebraic maps.

Conjecture 3.4.24. *A complex smooth projective variety X is special if and only if any two points of X are joined by a chain of entire curves. This is in turn equivalent to the apparently stronger property: any two points of X are contained in an entire curve.*

Conjecture 3.4.25. *Let c_X be the core map of a smooth projective complex variety X. Then, there exists a complex projective subvariety $W \subsetneq X$ such that any entire curve $h : \mathbb{C} \to X$ has image either contained in $c_X^{-1}(W)$, or in some fiber of c_X. Moreover, X is special if and only if it contains a dense entire curve.*

Proposition 3.4.26. *Assuming the conjectures above 3.4.21 and 3.4.25, the followings are equivalent:*

(i) there is an entire curve $h : \mathbb{C} \to X$;

(ii) $X(k')$ is infinite for some finite extension k'/k.

Proof. Assume $X(k')$ is infinite. Let Z be the Zariski closure of $X(k')$, then Z is positive dimensional and $Z(k')$ is dense in Z. Thus, by conjecture 3.4.21, Z is special. Apply then conjecture 3.4.25 to get the result.
Conversely, assume there is a dense entire curve $h : \mathbb{C} \to X$. Let Z be the Zariski closure of $h(\mathbb{C})$ and $Z' \to Z$ a resolution of singularities. Then h lifts to a Zariski dense entire curve in Z'. First, we assume that Z, and so Z', is defined over k. Thus, Z' is special by conjecture 3.4.25 and, by conjecture 3.4.21, there exists k'/k finite extension such that $Z'(k')$ is Zariski dense in Z'. Therefore $Z(k')$ is infinite and so is $X(k')$. In the general case, let Y be a resolution of the smallest subset of X defined over k and containing Z. If Y was not special, let $c_Y : Y \to C$ be its core map (which is defined over k). Then, $c \circ h(\mathbb{C})$ is contained in a strict algebraic subset $W \subsetneq C$ defined over k, thus $h(\mathbb{C}) \subseteq c_Y^{-1}(W) \subsetneq Y$ which contradicts minimality of Y. Therefore Y is special and by conjecture 3.4.21 there is a finite extension k'/k such that $Y(k')$ is infinite, whence the conclusion. qed

Remark. In the proof it is hidden also another equivalent property: X contains a special subvariety.

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